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INTERPOLATING SEQUENCES FOR QA_B

BY

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ABSTRACT. Let B be a closed algebra lying between H^∞ and L^∞ of the unit circle. We define $QA_B = H^\infty \cap \bar{B}$, the analytic functions in $Q_B = B \cap \bar{B}$. By work of Chang, Q_B is characterized by a vanishing mean oscillation condition. We characterize the sequences of points $\{z_n\}$ in the open unit disc for which the interpolation problem $f(z_n) = \lambda_n$, $n = 1, 2, \ldots$, is solvable with $f \in Q_B$ for any bounded sequence of numbers $\{\lambda_n\}$. Included as a necessary part of our proof is a study of the algebras QA_B and Q_B.

1. Introduction. Let H^∞ denote the Banach algebra of bounded analytic functions on the open unit disc $\mathbf{D} = \{z: |z| < 1\}$. Using radial limits we can identify H^∞ with a closed subalgebra of $L^\infty = L^\infty(\partial\mathbf{D})$. An H^∞ function can be recovered from its boundary values by means of the Poisson integral formula

$$f(z) = \int_{-\pi}^{\pi} f(e^{i\theta})\, dP_z(e^{i\theta}),$$

where

$$dP_z(e^{i\theta}) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta = P_z(e^{i\theta})\, d\theta.$$

We will also use this formula to define harmonic extensions of functions in $L^p = L^p(\partial\mathbf{D})$, $1 \leq p \leq \infty$.

An interpolating sequence is a sequence $\{z_n\} \subseteq \mathbf{D}$ with the property that for any bounded sequence of complex numbers $\{\lambda_n\}$ there exists $f \in H^\infty$ such that $f(z_n) = \lambda_n$ for all n. A well-known theorem of L. Carleson [1] states that a sequence $\{z_n\}$ is interpolating iff

$$\inf_n \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| > 0.$$

A Blaschke product

$$b(z) = \prod_n \frac{|z_n|(z_n - z)}{z_n(1 - \bar{z}_n z)}$$

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is called an interpolating Blaschke product if its zero set $\{z_n\}$ is an interpolating sequence. It is easy to check that

$$\prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| = |b'(z_n)| (1 - |z_n|^2),$$

hence a Blaschke product is interpolating precisely when $\inf_n |b'(z_n)| (1 - |z_n|^2) > 0$.

A function $f \in L^1(\partial \mathbf{D})$ is said to be in BMO if

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f - I(f)| \frac{d\theta}{2\pi} < \infty.$$

Here the supremum is taken over arcs $I \subseteq \partial \mathbf{D}$, $|I|$ denotes Lebesgue measure of I divided by 2π, and $I(f) = (1/|I|)\int_I f \, d\theta/2\pi$. It follows from results of P. Jones in [**14**] that if $\{z_n\} \subseteq \mathbf{D}$ is such that $\inf_n \prod_{m \neq n} |(z_m - z_n)/(1 - \bar{z}_m z_n)|$ is very close to 1 and $|\lambda_n| \leq 1$, then the interpolation problem $f(z_n) = \lambda_n$ can be solved by an H^∞ function whose boundary values have small BMO norm. In other words, "thinness" of a sequence implies interpolation with functions that oscillate very little. In this paper we prove an analogous result in which thinness of the entire sequence is replaced by thinness only in certain regions of the disc, and small BMO norm is replaced by small mean oscillation on certain arcs of $\partial \mathbf{D}$. Our result concerns function spaces arising in the theory of Douglas algebras, which we will now discuss briefly. For further information we suggest the reader consult [**3, 4, 9, 17, 19**, and **20**].

A *Douglas algebra* is a closed subalgebra of L^∞ containing H^∞. It is a consequence of the Gleason-Whitney Theorem [**11**] that the maximal ideal space $\mathfrak{M}(B)$ of a Douglas algebra B is naturally imbedded in $\mathfrak{M}(H^\infty)$, the maximal ideal space of H^∞. In [**3** and **17**], S.-Y. A. Chang and D. E. Marshall proved the following result, which had been conjectured by R. G. Douglas.

CHANG-MARSHALL THEOREM. *Every Douglas algebra is generated as a closed algebra over H^∞ by a family of complex conjugates of Blaschke products.*

In connection with this, we note that since a Blaschke product b is unimodular as an element of $L^\infty(\partial \mathbf{D})$, $\bar{b} = b^{-1}$ in $L^\infty(\partial \mathbf{D})$. An important part of Chang's proof is the study of a certain mean oscillation condition connected with a Douglas algebra. Let B be a Douglas algebra. A consequence of the Chang-Marshall Theorem is that if $U \subseteq \mathfrak{M}(H^\infty)$ is an open set containing $\mathfrak{M}(B)$, then $U \cap \mathbf{D}$ contains a set of the form

$$\{z \in \mathbf{D} : |b(z)| > \eta\},$$

where b is a Blaschke product in B^{-1} and $0 < \eta < 1$. Conversely, any set of this form is the intersection with $\mathbf{D}$ of a neighborhood of $\mathfrak{M}(B)$. Hence a statement such as "$\psi(z) \to 0$ as $z \to \mathfrak{M}(B)$" has the following obvious interpretation: given $\varepsilon > 0$ there is a Blaschke product $b \in B^{-1}$ and $0 < \eta < 1$ such that $|\psi(z)| < \varepsilon$ whenever $|b(z)| > \eta$. For a point $z \in \mathbf{D}$ we define $I_z \subseteq \partial \mathbf{D}$ to be the arc of length $2\pi(1 - |z|)$ centered at $z/|z|$. We now define

$Q_B = B \cap \bar{B}$, the largest C^*-algebra contained in B,

$QA_B = Q_B \cap H^\infty = \bar{B} \cap H^\infty$,

$\mathrm{VMO}_B = \{f \in \mathrm{BMO}: (1/|I_z|)\int_{I_z} |f - I_z(f)|\, d\theta/2\pi \to 0 \text{ as } z \to \mathfrak{M}(B)\}.$

We will also occasionally mention the space C_B, which is the C^*-algebra generated by the Blaschke products in B^{-1}. Among other things, Chang shows in [4] that $Q_B = \mathrm{VMO}_B \cap L^\infty$.

Before stating our result we need one more definition.

DEFINITION. A sequence $\{z_n\} \subseteq \mathbf{D}$ is *thin near* $\mathfrak{M}(B)$ if it is an interpolating sequence and

$$\prod_{m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| \to 1 \quad \text{as } z_n \to \mathfrak{M}(B).$$

By Theorem 4.3 of [12] an interpolating Blaschke product whose zero set misses some neighborhood of $\mathfrak{M}(B)$ is invertible in B. Using this fact it is easy to verify that if $\{z_n\}$ is an interpolating sequence with associated Blaschke product b, then $\{z_n\}$ is thin near $\mathfrak{M}(B)$ iff for any $0 < \eta < 1$ a factorization $b = b_1 b_2$ exists satisfying $b_1 \in B^{-1}$ and $|b_2'(z_n)|(1 - |z_n|^2) > \eta$ for all n such that $b_2(z_n) = 0$.

We can now state our main result.

THEOREM 1. *The following are equivalent for a sequence* $\{z_n\} \subseteq \mathbf{D}$:

(1) *For any bounded sequence of complex numbers* $\{\lambda_n\}$ *there exists* $f \in QA_B$ *such that* $f(z_n) = \lambda_n$ *for all* n.

(2) *For any bounded sequence of complex numbers* $\{\lambda_n\}$ *there exists* $f \in \mathrm{VMO}_B$ *such that* $f(z_n) = \lambda_n$ *for all* n.

(3) $\{z_n\}$ *is thin near* $\mathfrak{M}(B)$.

Moreover if condition (3) *is met we can find P. Beurling functions yielding* (1). *That is, there are functions* $\phi_n \in QA_B$ *such that* $\phi_n(z_k) = \delta_{nk}$ *and for any bounded sequences* $\{\lambda_n\}$, $\Sigma_n \lambda_n \phi_n \in QA_B$.

The proof of Theorem 1 will occupy the rest of this paper. The implication from (1) to (2) is of course trivial, and the implication from (2) to (3) is shown in §7. The main difficulty is in showing that (3) implies (1). This involves quite a few auxiliary results and is done in §§3–5.

We will now give a brief outline of the paper.

§2: This is a study of some basic facts about VMO_B.

§3: We assume the interpolating sequence z_n satisfies a certain technical condition called Λ_B. Let $\{\lambda_n\}$ be a bounded sequence and assume there is $g_0 \in Q_B$ such that $|g_0(z_n) - \lambda_n| \to 0$ as $z_n \to \mathfrak{M}(B)$. We perturb g_0 to obtain a function $g \in C^\infty(\mathbf{D})$ such that $g(z) = \lambda_n$ when $|(z - z_n)/(1 - \bar{z}_n z)| < \eta$, where η is a small number, and such that the measures $|\nabla g(z)|^2(1 - |z|^2)\, dx\, dy$ and $|\Delta g(z)|(1 - |z|^2)\, dx\, dy$ satisfy a condition which we call a B-Carleson condition. Let b be the Blaschke product with zeros $\{z_n\}$. Using results of §2 and the condition Λ_B we show that there is $q \in QA_B$ such that $qb \in QA_B$, and such that $|\nabla g|^2(1 - |z|^2)\, dx\, dy/|q|^2$ and $|\Delta g|(1 - |z|^2)\, dx\, dy/|q|$ are still B-Carleson measures. We now set $f = g + qb\alpha$, where α is to be chosen so that $f \in QA_B$. This is done by solving the $\bar{\partial}$-equation $\partial\alpha/\partial\bar{z} = -(\partial g/\partial\bar{z})/qb$; results in §2 imply that this equation can be solved by a function α with boundary values in Q_B. Together with the above condition on ∇g, this implies that $f \in QA_B$. Clearly $f(z_n) = \lambda_n$ for all n.

§4: Using a construction due to J. Garnett and P. Jones we show that if $\{z_n\}$ is thin near $\mathfrak{M}_B$, then $\{z_n\}$ satisfies the condition Λ_B.

§5: We complete the proof that (3) implies (1) by showing that for any bounded sequence $\{\lambda_n\}$ there is $g_0 \in Q_B$ such that $|g_0(z_n) - \lambda_n| \to 0$ as $z_n \to \mathfrak{M}(B)$. This is done by an explicit construction related to, but somewhat easier than, the Garnett-Jones construction.

§6: Using a method due to N. Th. Varopoulos we construct the P. Beurling functions which give (1). In order to assure that the desired linear combinations of these functions are in QA_B it is necessary that the estimates in the preceding sections depend only on the sequence $\{z_n\}$; this requirement unfortunately forces all our proofs to be more complicated.

§7: This is the proof that (2) implies (3).

REMARKS. (1) If the interpolating sequence $\{z_n\}$ is such that its associated Blaschke product is in B^{-1}, then certainly $\{z_n\}$ is thin near $\mathfrak{M}(B)$. In this case the proof of the H^∞ interpolation theorem due to J. P. Earl **[5]** yields an interpolating function in $C_B \cap H^\infty$.

(2) The existence of P. Beurling functions solving the H^∞ interpolation problem is shown in **[2]**. Jones, in **[15]**, has obtained explicit formulas for such functions. These formulas as written do not answer the present question, but it seems possible that some alteration of them might yield our results.

(3) Because of the quantitative nature of our methods we actually show a stronger result than (3) implies (1). The alert and patient reader will be able to see that our methods establish the following

THEOREM 2. *Let $\{z_n\} \subseteq \mathbf{D}$ be an interpolating sequence. Then there exist functions $\phi_n \in H^\infty$ such that $\phi_n(z_k) = \delta_{nk}, \Sigma_n \lambda_n \phi_n \in H^\infty$ for any bounded sequence $\{\lambda_n\}$, and such that the following statements are true. Let $\varepsilon > 0$ be given. There then exists $0 < \eta < 1$ such that if a Blaschke product b and a number $0 < \rho < 1$ are such that $\Pi_{m \neq n} |(z_m - z_n)/(1 - \bar{z}_m z_n)| > \eta$ whenever $|b(z_n)| > \rho$, then*

$$\frac{1}{|I_z|} \int_{I_z} \left| \sum_n \lambda_n \phi_n - I_z\left(\sum_n \lambda_n \phi_n \right) \right| \frac{d\theta}{2\pi} < \varepsilon$$

whenever $|b(z)| > \rho'$, if $\{\lambda_n\}$ is any sequence for which $|\lambda_n| \leq 1$ for all n; here $0 < \rho' < 1$ depends only on ε and ρ.

In particular, the result mentioned at the beginning of this section about interpolating with functions of small BMO norm follows from our proof. An explicit proof of Theorem 2 would seem to be too cumbersome to write down.

We now list some notations that will be used throughout this paper.

DEFINITIONS. The letters C, C', C_1, etc. will denote constants, not necessarily the same at each occurrence.

If $I = \{e^{i\theta}: \alpha \leq \theta \leq \alpha + l\}$ is an arc, then $S_I = \{re^{i\theta}: e^{i\theta} \in I, 1 - l/2\pi \leq r < 1\}$ and $T_I = \{re^{i\theta} \in S_I: 1 - l/2\pi \leq r \leq 1 - \frac{1}{2}l/2\pi\}$; i.e., T_I is the "top half" of S_I.

If $z \in \mathbf{D}$, then I_z is the arc of length $2(1 - |z|)$ centered at $z/|z|$; we then denote by S_z and T_z respectively the sets S_{I_z} and T_{I_z}. If I is an arc then z_I is the point in $\mathbf{D}$ such that $I = I_{z_I}$.

If $z, w \in \mathbf{D}$ the pseudo-hyperbolic distance between them is $\rho(z, w) = |(z-w)/(1-\bar{z}w)|$. For $a \in \mathbf{D}$ we denote by L_a the linear fractional map $L_a(z) = (z+a)/(1+\bar{a}z)$. It is well known that $\rho(L_a(z), L_a(w)) = \rho(z, w)$.

If I is an arc and q is a positive integer, then qI is the arc with the same center as I and length q times that of I (if q times the length of I is greater than 2π, set $qI = \partial\mathbf{D}$). We will also use the notations $\tilde{I} = 3I$ and $\tilde{\tilde{I}} = 5I$.

A *dyadic arc* is an arc of the form $\{e^{i\theta}: 2\pi k/2^n \leqslant \theta \leqslant 2\pi(k+1)/2^n\}$ for $n \geqslant 0$ and $0 \leqslant k \leqslant 2^n - 1$.

If $E \subseteq \partial\mathbf{D}$ we denote by $|E|$ the Lebesgue measure of E divided by 2π. If I is an arc and f is a function on $\partial\mathbf{D}$, then

$$I(f) = \frac{1}{|I|}\int_I f\frac{d\theta}{2\pi},$$

$$M_I(f) = \frac{1}{|I|}\int_I |f - I(f)|\frac{d\theta}{2\pi},$$

$$V_I(f) = \sup\{|f(e^{i\theta_1}) - f(e^{i\theta_2})| : e^{i\theta_1}, e^{i\theta_2} \in I\}.$$

Thus $\|f\|_* = \sup_I M_I(f)$.

A *Carleson measure* is a measure μ on $\mathbf{D}$ for which

$$\|\mu\|_* = \sup\{|\mu|(S_I)/|I| : I \text{ an arc}\} < \infty.$$

It is well known (see Chapter 6 of [**9**]) that the norm $\|\mu\|_*$ is equivalent to the norm $\sup\{\int(1-|z|^2)/|1-\bar{\zeta}z|^2\, d|\mu|(\zeta): z \in \mathbf{D}\}$.

We will denote by H^p, $1 \leqslant p \leqslant \infty$, the usual Hardy spaces of analytic functions, and set $H_0^p = \{f \in H^p: f(0) = 0\}$. We will write L^p for $L^p(\partial\mathbf{D})$. The orthogonal projections of L^2 onto H^2 and $(H^2)^\perp = \overline{H_0^2}$ will be denoted respectively by P and Q. For $f \in L^2$, we will denote the harmonic conjugate (Hilbert transform) of f by $\tilde{f}$.

Finally, we state a well-known consequence of Hall's Lemma that we will use repeatedly. For a proof see Chapter 8 of [**9**].

LEMMA 1.1. *If $0 < \eta < 1$ and $\varepsilon > 0$, then there is $0 < \kappa < 1$ such that if $a \in \mathbf{D}$ and $f \in H^\infty$, $\|f\|_\infty \leqslant 1$, is such that there exists $z \in T_a$ satisfying $|b(z)| > \kappa$, then the set $\{w \in S_a: |b(w)| < \eta\}$ is contained in a union of squares $S_{w_j} \subset S_a$ with $\sum_j(1-|w_j|) < \varepsilon(1-|a|)$.*

2. Basic facts about VMO_B. In this section we will study analogues for VMO_B of various well-known facts about BMO. The main result is Theorem 2.14.

THEOREM 2.1. *Let $f \in \mathrm{BMO}$, $\|f\|_* \leqslant 1$, and let $1 < p < \infty$. Then the following are equivalent*:

(i) $f \in \mathrm{VMO}_B$,

(ii) $(1/|I_z|)\int_{I_z}|f - I_z(f)|\, d\theta/2\pi \to 0$ *as* $z \to \mathfrak{M}(B)$,

(iii) $((1/|I_z|)\int_{I_z}|f - I_z(f)|^p\, d\theta/2\pi)^{1/p} \to 0$ *as* $z \to \mathfrak{M}(B)$,

(iv) $\int|f - f(z)|\, dP_z \to 0$ *as* $z \to \mathfrak{M}(B)$,

(v) $(\int|f - f(z)|^p\, dP_z)^{1/p} \to 0$ *as* $z \to \mathfrak{M}(B)$,

(vi) $\int(1-|z|^2)/(|1-\bar{\zeta}z|^2)|\nabla f(\zeta)|^2(1-|\zeta|^2)\, d\xi\, d\eta \to 0$ *as* $z \to \mathfrak{M}(B)$, *where $\zeta = \xi + i\eta$ and ∇f denotes the gradient of the harmonic extension of f.*

REMARK. This theorem is mostly known, but we will need more precise information than existing proofs seem to give. Our proof will establish, for instance, that there is a constant C depending only on p such that if $\varepsilon > 0, \|f\|_* \leq 1$, b is a Blaschke product, and $0 < \eta < 1$ are such that $(1/|I_z|)\int_{I_z}|f - I_z(f)|\,d\theta/2\pi < \varepsilon$ whenever $|b(z)| > \eta$, then $((1/|I_z|)\int_{I_z}|f - I_z(f)|^p\,d\theta/2\pi)^{1/p} < C\varepsilon$ whenever $|b(z)| > \eta'$, where $0 < \eta' < 1$ depends only on ε, p, and η. The other implications in the theorem can be similarly rephrased.

PROOF OF THEOREM 2.1. (i) $\Leftrightarrow$ (ii) is the definition of VMO_B, and (iii) $\Rightarrow$ (ii), (v) $\Rightarrow$ (iv) are immediate consequences of Hölder's inequality. Once we have established the equivalences of (ii)–(v), (vi) can be proven equivalent to the others by showing it to be equivalent to (v) for the case $p = 2$. This latter equivalence is shown by Chang in [**3** and **4**]; we now sketch her argument for the sake of completeness. A calculation based on Fourier series establishes that

$$\frac{1}{2}\int |g - g(0)|^2 \frac{d\theta}{2\pi} \leq \frac{1}{2\pi}\int_{\mathbf{D}}\int |\nabla g(\zeta)|^2 \left(1 - |\zeta|^2\right) d\xi\, d\eta$$

$$\leq \int |g - g(0)|^2 \frac{d\theta}{2\pi}$$

for any $g \in L^2$; replacing g by $f \circ L_z$ yields

$$\frac{1}{2}\int |f - f(z)|^2\, dP_z \leq \frac{1}{2\pi}\int_{\mathbf{D}}\int \frac{1 - |z|^2}{|1 - \bar{\zeta} z|^2} |\nabla f(\zeta)|^2 \left(1 - |\zeta|^2\right) d\xi\, d\eta$$

$$\leq \int |f - f(z)|^2\, dP_z,$$

which easily gives the desired equivalence.

It remains to show that (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v). In the remainder of the proof, C will denote a constant depending at most only on p, and not necessarily the same at each occurrence.

(iv) $\Rightarrow$ (ii): Let $\varepsilon > 0$. Choose $b \in B^{-1}$ and $0 < \eta < 1$ such that $\int |f - f(z)|\, dP_z < \varepsilon$ whenever $|b(z)| > \eta$. We have $P_z(e^{i\theta}) > C/|I_z|$ if $e^{i\theta} \in I_z$. Hence if $|b(z)| > \eta$,

$$\frac{1}{|I_z|}\int_{I_z} |f - f(z)| \frac{d\theta}{2\pi} < \frac{1}{C}\int_{I_z} |f - f(z)|\, dP_z.$$

This easily gives $(1/|I_z|)\int_{I_z}|f - I_z(f)|\,d\theta/2\pi < 2\varepsilon/C$.

(iii) $\Rightarrow$ (iv): The proof is an adaptation of the argument used to establish the analogous BMO result in [**19**, Chapter 5]. Let $\varepsilon > 0$ and choose $b \in B^{-1}$ and $0 < \eta < 1$ such that $((1/|I_z|)\int |f - I_z(f)|^p\, d\theta/2\pi)^{1/p} < \varepsilon$ whenever $|b(z)| > \eta$. Set $I_n = 2^n I_z$ for $n = 0, 1, \ldots, N-1$, where N is the smallest integer such that $2^N|I_z| \geq 1$, and set $I_N = \partial\mathbf{D}$. Now $P_z(e^{i\theta}) \leq C/|I_z|$ for all $e^{i\theta}$, and $P_z(e^{i\theta}) \leq C/2^{2n}|I_z|$ for $e^{i\theta} \notin I_n$. Hence

$$\int |f - I_z(f)|^p\, dP_z = \int_{I_z} |f - I_z(f)|^p\, dP_z + \sum_{n=0}^{N-1} \int_{I_{n+1}\setminus I_n} |f - I_z(f)|^p\, dP_z$$

$$\leq C\frac{1}{|I_z|}\int |f - I_z(f)|^p \frac{d\theta}{2\pi} + C\sum_{n=0}^{N-1} \frac{1}{2^{n-1}}\frac{1}{|I_{n+1}|}\int_{I_{n+1}} |f - I_z(f)|^p\, dP_z.$$

Now we note that if I is any arc then

$$|(2I)(f)-I(f)|\leq\frac{1}{|I|}\int_I|f-(2I)(f)|\frac{d\theta}{2\pi}\leq 2M_{2I}(f).$$

Hence $|I_n(f)-I_z(f)|\leq 2\Sigma_{j=1}^n M_{I_j}(f)$, and so

$$\left(\frac{1}{|I_{n+1}|}\int_{I_{n+1}}|f-I_z(f)|^p\frac{d\theta}{2\pi}\right)^{1/p}\leq\frac{1}{|I_{n+1}|}\int_{I_{n+1}}|f-I_{n+1}(f)|^p\frac{d\theta}{2\pi}+2\sum_{j=1}^{n+1}M_{I_j}(f).$$

Since $\|f\|_*\leq 1$, we have $M_{I_j}(f)\leq 1$ for all j. It also follows from the John-Nirenberg Theorem **[13]** that

$$\left(\frac{1}{|I_{n+1}|}\int_{I_{n+1}}|f-I_{n+1}(f)|^p\frac{d\theta}{2\pi}\right)^{1/p}<C.$$

Choose K so high that
(2.2)

$$\sum_{n=K+1}^{N-1}\frac{1}{2^{n-1}}\frac{1}{|I_{n+1}|}\int_{I_{n+1}}|f-I_z(f)|^p\frac{d\theta}{2\pi}\leq\sum_{n=K+1}^{\infty}\frac{1}{2^{n-1}}(C+2(n+1))^p<\varepsilon^p.$$

Using Schwarz's Lemma, choose $0<\kappa<1$ so that $|b(z)|>\kappa$ implies that $|b(w)|>\eta$ if $w\in T_{I_j}$ for any $j\leq K+1$. Then using

$$M_{I_j}(f)\leq\left(\frac{1}{|I_j|}\int_{I_j}|f-I_j(f)|^p\frac{d\theta}{2\pi}\right)^{1/p},$$

we have

$$\text{(2.3)}\quad\sum_{n=0}^{K}\frac{1}{2^{n-1}}\frac{1}{|I_{n+1}|}\int_{I_{n+1}}|f-I_z(f)|^p\frac{d\theta}{2\pi}<\sum_{n=0}^{K}\frac{1}{2^{n-1}}[\varepsilon+2(n+1)\varepsilon]^p<C\varepsilon^p.$$

Combining (2.2) and (2.3) with the fact that $((1/|I_z|)\int_{I_z}|f-I_z(f)|^p\,d\theta/2\pi)^{1/p}<\varepsilon$, we see that $(\int|f-I_z(f)|^p\,dP_z)^{1/p}<C\varepsilon$ if $|b(z)|>\kappa$. Hence $(\int|f-f(z)|^p\,dP_z)^{1/p}<2C\varepsilon$ for such z.

(ii) ⇒ (iii): Our proof is similar to the proof of the John-Nirenberg Theorem **[13]**. Let $\varepsilon>0$ and let a Blaschke product $b\in B^{-1}$ and $0<\eta<1$ be such that $M_{I_z}(f)<\varepsilon$ whenever $|b(z)|>\eta$. We first note that if I is an arc and $E\subseteq I$, then

$$\left(\frac{1}{|I|}\int_E|f-I(f)|^p\frac{d\theta}{2\pi}\right)^{1/p}\leq\left(\frac{1}{|I|}\int_I|f-I(f)|^{2p}\frac{d\theta}{2\pi}\right)^{1/2p}\left(\frac{|E|}{|I|}\right)^{1/2p}$$
$$<C(|E|/|I|)^{1/2p},$$

where the last inequality follows from the John-Nirenberg Theorem and the fact that $\|f\|_*\leq 1$. Choose $\gamma>0$ so that if $|E|<\gamma|I|$, this last quantity is less than ε.

For an arc I denote by $\mathfrak{D}(I)$ the dyadic decomposition of I, i.e., the collection of arcs obtained from I by successive halvings. Using Lemma 1.1, choose $0<\kappa<1$ such that $|b(z)|>\kappa$ implies

$$\left|\bigcup\{I\in\mathfrak{D}(I_z):\exists w\in T_I\text{ such that }|b(w)|\leq\eta\}\right|<\gamma|I_z|.$$

Now let $z \in \mathbf{D}$ satisfy $|b(z)| > \kappa$. Let

$$\{J_l\} = \{I \in \mathfrak{D}(I_z) \colon \exists w \in T_I \text{ such that } |b(w)| \leqslant \eta\}.$$

We note that our choice of γ and κ guarantee that

$$\left(\frac{1}{|I_z|}\int_{\cup J_l}|f - I_z(f)|^p \frac{d\theta}{2\pi}\right)^{1/p} < \varepsilon.$$

We want to estimate the measure of $E_n = \{e^{i\theta} \in I \setminus \cup J_l \colon |f(e^{i\theta}) - I_z(f)| > 4n\varepsilon\}$. To this end, set $I = I_z$ and denote by $\{I_{k_1}\}_{k_1}$ the maximal arcs in $\mathfrak{D}(I)$ satisfying $I_{k_1} \not\subseteq J_l$ for any l and $(1/|I_{k_1}|)\int_{I_{k_1}}|f - I(f)|\, d\theta/2\pi \geqslant 2\varepsilon$. Then by maximality, $(1/|I_{k_1}|)\int_{I_{k_1}}|f - I(f)|\, d\theta/2\pi < 4\varepsilon$. For each k_1, denote by $\{I_{k_1k_2}\}_{k_2}$ the maximal arcs in $\mathfrak{D}(I_{k_1})$ satisfying $I_{k_1k_2} \not\subseteq J_l$ for any l and $(1/|I_{k_1k_2}|)\int_{I_{k_1k_2}}|f - I_{k_1}(f)|\, d\theta/2\pi \geqslant 2\varepsilon$. Continuing this process, we obtain a set of arcs $\{I_{k_1\cdots k_n}\}$. Now the inequalities

$$\frac{1}{|I_{k_1\cdots k_j}|}\int_{I_{k_1\cdots k_j}}\left|f - I_{k_1\cdots k_{j-1}}(f)\right|\frac{d\theta}{2\pi} < 4\varepsilon, \qquad j = 1,\ldots,n,$$

together with Lebesgue's differentiation theorem imply that, except for a set of measure zero, $E_n \subset \cup_{k_1,\ldots,k_n} I_{k_1\cdots k_n}$. Since $I_{k_1\cdots k_j} \not\subseteq J_l$ for any l, we have $M_{I_{k_1\cdots k_{j-1}}}(f) < \varepsilon$, hence

$$\varepsilon|I_{k_1\cdots k_{j-1}}| > \int_{I_{k_1\cdots k_{j-1}}}|f - I_{k_1\cdots k_{j-1}}(f)|\frac{d\theta}{2\pi}$$

$$\geqslant \sum_{k_j}\int_{I_{k_1\cdots k_j}}|f - I_{k_1\cdots k_{j-1}}(f)|\frac{d\theta}{2\pi} \geqslant 2\varepsilon\sum_{k_j}|I_{k_1\cdots k_j}|.$$

Iterating this inequality, we obtain $\sum_{k_1,\ldots,k_n}|I_{k_1\cdots k_n}| < |I|/2^n$, hence $|E_n| < |I|/2^n$. This easily implies that if $F_\alpha = \{e^{i\theta} \in I \setminus \cup J_l \colon |f(e^{i\theta}) - I(f)| > \alpha\}$, then $|F_\alpha| < 2|I|/2^{\alpha/4\varepsilon}$ for $\alpha \geqslant 4\varepsilon$. This yields

$$\frac{1}{|I|}\int_I|f - I(f)|^p\frac{d\theta}{2\pi} = \frac{1}{|I|}\int_{I\setminus\cup J_l}|f - I(f)|^p\frac{d\theta}{2\pi} + \frac{1}{|I|}\int_{\cup J_l}|f - I(f)|^p\frac{d\theta}{2\pi}$$

$$< \frac{1}{|I|}\int_0^\infty\left|\{e^{i\theta} \in I \setminus \cup J_l \colon |f(e^{i\theta}) - I(f)| > \alpha\}\right| p\alpha^{p-1}\, d\alpha + \varepsilon^p$$

$$\leqslant \frac{1}{|I|}\int_0^{4\varepsilon}|I|p\alpha^{p-1}\, d\alpha + \frac{1}{|I|}\int_{4\varepsilon}^\infty\frac{2}{2^{\alpha/4\varepsilon}}|I|p\alpha^{p-1}\, d\alpha + \varepsilon^p$$

$$< (4\varepsilon)^p + 2p\left(\frac{4}{\log 2}\right)^p\Gamma(p)\varepsilon^p + \varepsilon^p.$$

Thus $((1/|I_z|)\int_{I_z}|f - I_z(f)|^p\, d\theta/2\pi)^{1/p} < C\varepsilon$ if $|b(z)| > \kappa$, as desired. This completes the proof of the theorem. □

COROLLARY 2.4. *Let $f \in \mathrm{VMO}_B$. Then $|f(z) - I_z(f)| \to 0$ as $z \to \mathfrak{M}(B)$. More precisely, given $\varepsilon > 0$ there exists $\varepsilon' > 0$ making the following statement true. If $f \in \mathrm{BMO}$ with $\|f\|_* \leqslant 1$ and a Blaschke product b and a number $0 < \eta < 1$ are such that $M_{I_z}(f) < \varepsilon'$ whenever $|b(z)| > \eta$, then $|f(z) - I_z(f)| < \varepsilon$ whenever $|b(z)| > \kappa$, where $0 < \kappa < 1$ depends only on ε and η.*

Proof. The proof that (iii) implies (iv) of course also works for $p = 1$, and this proof actually establishes a bound on $\int |f - I_z(f)|\, dP_z \geq |f(z) - I_z(f)|$. $\square$

Recall that P, Q denote respectively the orthogonal projections of L^2 onto H^2 and $(H^2)^\perp = \overline{H}_0^2$. It is implicit in the work of Chang that $B = \{f \in L^\infty\colon Qf \in \mathrm{VMO}_B\}$, hence in particular that this latter set is an algebra. Our next result provides a direct proof of this fact.

Theorem 2.5. *Let $f, g \in L^\infty$ with $\|f\|_\infty \leq 1, \|g\| \leq 1$, let $\varepsilon > 0$, let b be a Blaschke product, and let $0 < \eta < 1$. Say that $(\int |Qf - Qf(z)|^2\, dP_z)^{1/2} < \varepsilon$ and $(\int |Qg - Qg(z)|^2\, dP_z)^{1/2} < \varepsilon$ when $|b(z)| > \eta$. Then there is $0 < \kappa < 1$ such that*

$$\left(\int |Q(fg) - Q(fg)(z)|^2\, dP_z\right)^{1/2} < C\varepsilon$$

whenever $|b(z)| > \kappa$, where κ depends only on ε and η, and C is a universal constant.

Proof. For $z \in \mathbf{D}$ define $H_z^2 = \{f \in H^2\colon f(z) = 0\}$. We have

$$\begin{aligned}\int |Q(fg) - Q(fg)(z)|^2\, dP_z &\\ &= \sup\left\{\left|\int [Q(fg) - Q(fg)(z)]h\, dP_z\right| : h \in L^2, \int |h|^2\, dP_z \leq 1\right\}\\ &= \sup\left\{\left|\int [Q(fg) - Q(fg)(z)]h\, dP_z\right| : h \in H_z^2, \int |h|^2\, dP_z \leq 1\right\}\\ &= \sup\left\{\left|\int Q(fg)\cdot h\, dP_z\right| : h \in H_z^2, \int |h|^2\, dP_z \leq 1\right\}\end{aligned}$$

since $\int h\, dP_z = h(z) = 0$ for $h \in H_z^2$. For brevity we write $\sup\{\cdot\}$ for

$$\sup\left\{\cdot : h \in H_z^2, \int |h|^2\, dP_z \leq 1\right\}.$$

Continuing our chain of equalities:

$$\begin{aligned}\sup\left\{\left|\int Q(fg)\cdot h\, dP_z\right|\right\} &= \sup\left\{\left|\int fgh\, dP_z\right|\right\}\\ &\leq \sup\left\{\left|\int [f - f(z)]gh\, dP_z\right|\right\} + \sup\left\{|f(z)|\left|\int gh\, dP_z\right|\right\}\\ &\leq \sup\left\{\left|\int [Qf - Qf(z)]gh\, dP_z\right|\right\} + \sup\left\{\left|\int [Pf - Pf(z)]gh\, dP_z\right|\right\}\\ &\quad + \sup\left\{\left|\int gh\, dP_z\right|\right\}.\end{aligned}$$

Now $|\int [Qf - Qf(z)]gh\, dP_z| \leq (\int |Qf - Qf(z)|^2\, dP_z)^{1/2} < \varepsilon$ and

$$\left|\int gh\, dP_z\right| = \left|\int [Qg - Qg(z)]h\, dP_z\right| \leq \left(\int |Qg - Qg(z)|^2\, dP_z\right)^{1/2} < \varepsilon$$

if $|b(z)| > \eta$. The middle integral can be rewritten as

$$\int [Pf - Pf(z)][Qg - Qg(z)]h\, dP_z.$$

By the extended Hölder inequality this is bounded by

$$\left(\int |Pf - Pf(z)|^4\, dP_z\right)^{1/4}\left(\int |Qf - Qf(z)|^4\, dP_z\right)^{1/4}\left(\int |h|^2\, dP_z\right)^{1/2}.$$

The third factor is of course bounded by 1. The analogue for BMO of the implication from (ii) to (v) in Theorem 2.1 is well known—see Chapter 4 of **[19]**—and together with the boundedness of the projection P on BMO, implies that the first factor is bounded by a constant C_1. By Theorem 2.1 there is κ such that $\eta < \kappa < 1$ and C_2 such that if $|b(z)| > \kappa$, then the second factor is bounded by $C_2\varepsilon$. Hence if $|b(z)| > \kappa$, then

$$\left(\int |Q(fg) - Q(fg)(z)|^2\, dP_z\right)^{1/2} < C_3\varepsilon. \quad \square$$

COROLLARY 2.6. *Let N be a positive integer and let $f_1, \ldots, f_n \in L^\infty$ with $\|f_j\|_\infty \leq 1$. Suppose b is a Blaschke product and $0 < \eta < 1$, $\varepsilon > 0$ are such that*

$$\left(\int |Qf_j - Qf_j(z)|^2\, dP_z\right)^{1/2} < \varepsilon$$

for $j = 1, \ldots, N$ whenever $|b(z)| > \eta$. Then there is $0 < \kappa < 1$ depending only on ε, η, and N, and there is C_N depending only on N such that

$$\left(\int |Q(f_1 \cdots f_N) - Q(f_1 \cdots f_N)(z)|^2\, dP_z\right)^{1/2} < C_N\varepsilon$$

whenever $|b(z)| > \kappa$.

PROOF. Induction on Theorem 2.5. $\square$

In **[4]**, Chang shows that $\mathrm{VMO}_B = C_B + \tilde{C}_B$. Of course this implies that $\mathrm{VMO}_B = Q_B + \tilde{Q}_B$. Our next result shows that this decomposition can be done in a uniform way.

THEOREM 2.7. *Let $0 < C_0 \leq 1$ be a number such that if $f \in \mathrm{BMO}$ with $f(0) = 0$ and $\|f\|_* \leq C_0$, then f can be written as $f = f_1 + \tilde{f}_2$ where $\|f_1\|_\infty \leq 1$, $\|f_2\|_\infty \leq 1$ (such a number exists by C. Fefferman's Duality Theorem* **[6, 7]**). *Then if $\|f\|_* \leq C_0$ and $f(0) = 0$ we can write $f = f_1 + \tilde{f}_2$ with $\|f_1\|_\infty \leq 6$, $\|f_2\|_\infty \leq 6$ and so that the following will be true. Given $\varepsilon > 0$ there exists $\varepsilon' > 0$ depending only on ε such that if $(\int |f - f(z)|^2\, dP_z)^{1/2} < \varepsilon'$ whenever $|b(z)| > \eta$, where b is some Blaschke product and $0 < \eta < 1$, then $(\int |f_1 - f_1(z)|^2\, dP_z)^{1/2} < \varepsilon$ and $(\int |f_2 - f_2(z)|^2\, dP_z)^{1/2} < \varepsilon$ whenever $|b(z)| > \eta'$, where $0 < \eta' < 1$ depends only on ε and η.*

PROOF. Our proof is a combination of the proof of the decomposition $\mathrm{VMO}_B = C_B + \tilde{C}_B$ in **[9**, Chapter 9] with our Corollary 2.6. Assume f is real and write $f = v + \tilde{w}$, where $\|v\|_\infty \leq 1$ and $\|w\|_\infty \leq 1$, and define $g = \frac{1}{3}(v + iw)$. Since $v + \tilde{w} = v + iw + \tilde{w} - iw$, we have $Qg = 3Qf$. By Nevanlinna's Theorem **[9**, Theorem 4.3, Chapter 4], there is a unimodular function u such that $g = u - h$ for $h \in H^\infty$ and $d(u, H_0^\infty) = 1$. Since $\|u - h\|_\infty < \frac{2}{3}$ it follows as in the proof of [**9**, Lemma 4.3, Chapter 9], that $|h(z)| > \frac{1}{3}$ for all $z \in \mathbf{D}$. We have $\|1 - \bar{u}h\|_\infty < \frac{2}{3}$, so

$$\bar{u}h \in S_1 = \left\{re^{i\theta}: \tfrac{1}{3} \leq r \leq \tfrac{5}{3}, |\theta| \leq \sin^{-1}\tfrac{2}{3}\right\}.$$

Therefore

$$uh^{-1} = (\bar{u}h)^{-1} \in S_2 = \left\{re^{i\theta}: \tfrac{3}{5} \leq r \leq 3, |\theta| \leq \sin^{-1}\tfrac{2}{3}\right\}.$$

So $\|1 - \frac{1}{10}uh^{-1}\|_\infty < \frac{99}{100}$, hence we can write $10\bar{u}h = \sum_{n=0}^\infty (1 - \frac{1}{10}uh^{-1})^n$, or $\bar{u} = (10h)^{-1}\sum_{n=0}^\infty(1 - \frac{1}{10}uh^{-1})^n$. Now let $\varepsilon > 0$ be given and choose N so that $\frac{3}{10}\sum_{n=N+1}^\infty(\frac{99}{100})^n < \frac{1}{4}\varepsilon$. We now have

$$\frac{1}{10h}\sum_{n=0}^{N}\left(1 - \frac{1}{10}uh^{-1}\right)^n = \sum_{n=0}^{N} c_{nN}\left(\frac{1}{10h}\right)^{n+1} u^n,$$

so

$$\left(\int |Q\bar{u} - Q\bar{u}(z)|^2\, dP_z\right)^{1/2}$$

$$\leq \sum_{n=0}^{N} |c_{nN}| \left(\int \left| Q\left(\left(\frac{1}{10h}\right)^{n+1} u^n\right) - Q\left(\left(\frac{1}{10h}\right)^{n+1} u^n\right)(z)\right|^2 dP_z\right)^{1/2} + \frac{\varepsilon}{4}.$$

Let $\varepsilon' > 0$ be very small and suppose that a Blaschke product b and $0 < \eta < 1$ are such that $(\int |Qf - Qf(z)|^2\, dP_z)^{1/2} < \varepsilon'$ when $|b(z)| > \eta$. Since $Qu = Qg = 3Qf$, if ε' is small enough we can find by Corollary 2.6 a number η', $\eta < \eta' < 1$, such that if $|b(z)| > \eta'$ then

$$\sum_{n=0}^{N} |c_{nN}| \left(\int \left| Q\left(\left(\frac{1}{10h}\right)^{n+1} u^n\right) - Q\left(\left(\frac{1}{10h}\right)^{n+1} u^n\right)(z)\right|^2 dP_z\right)^{1/2} < \frac{\varepsilon}{4}.$$

Thus $(\int |Q\bar{u} - Q\bar{u}(z)|^2\, dP_z)^{1/2} < \varepsilon/2$ if $|b(z)| > \eta'$. Since $Q\bar{u} = \overline{Pu - Pu(0)}$, this shows that $(\int |u - u(z)|^2\, dP_z)^{1/2} < \varepsilon$ if $|b(z)| > \eta'$.

Now since $k - i\tilde{k} = 0$ for any analytic k and $f = 3u - 3h + \tilde{w} - iw$, we have $f - i\tilde{f} = 3u - i3\tilde{u}$. Thus $f = \operatorname{Re}(f - i\tilde{f}) = \operatorname{Re}(3u - i3\tilde{u})$, and since

$$\left(\int |\tilde{u} - \tilde{u}(z)|^2\, dP_z\right)^{1/2} = \left(\int |u - u(z)|^2\, dP_z\right)^{1/2}$$

for any $z \in \mathbf{D}$ our proof can be completed by setting $f_1 = 3\operatorname{Re} u$, $f_2 = 3\operatorname{Im} u$. □

We next give a method (Theorem 2.14) for solving certain $\bar{\partial}$-equations with boundary values in Q_B.

DEFINITION. Suppose F is a function on $\mathbf{D}$ and u is a function on $\partial\mathbf{D}$. We say F has *L^1 boundary function u* if $\lim_{r\to 1}\int |F(re^{i\theta}) - u(e^{i\theta})|\, d\theta/2\pi = 0$.

THEOREM 2.8. *Let $\Phi_1, \Phi_2\colon \mathbf{D} \to \mathbf{R}^+$ satisfy $0 \leq \Phi_1, \Phi_2 \leq M$ and let g be a function on $\mathbf{D}$ such that*

$$\int\!\!\int_{\mathbf{D}} |g \circ L_a|^2 |L_a'|^2 (1 - |z|^2)\, dx\, dy \leq \Phi_1(a)$$

and

$$\int_{\mathbf{D}}\!\!\int \left|\frac{\partial g}{\partial z}\circ L_a\right| |L_a'|^2 (1-|z|^2)\,dx\,dy \leqslant \Phi_2(a).$$

Then there is a function F on $\mathbf{D}$ with $\partial F/\partial \bar z = g$, having an L^1 boundary function u for which $\int |u - \int u\,dP_a|^2\,dP_a \leqslant C(\Phi_1(a) + M\Phi_2(a))$ for all $a \in \mathbf{D}$.

Proof. The conditions imply that $|g\circ L_a|^2|L_a'|^2(1-|z|^2)\,dx\,dy$ and $|\partial g/\partial z \circ L_a||L_a'|^2(1-|z|^2)\,dx\,dy$ are Carleson measures with Carleson norms at most CM. Define

$$d\mu_g = |g|^2|z|\log(1/|z|)\,dx\,dy$$

and

$$d\nu_g = |\partial g/\partial z||z|\log(1/|z|)\,dx\,dy;$$

μ_g and ν_g are then Carleson so $\partial U/\partial \bar z = g$ has a solution and any solution U must satisfy

$$\lim_{r\to 1}\int U(re^{i\theta})h(re^{i\theta})\frac{d\theta}{2\pi} = \frac{2}{\pi}\int_{\mathbf{D}}\!\!\int\left(h'g + h\frac{\partial g}{\partial z}\right)\log\frac{1}{|z|}\,dx\,dy \tag{2.9}$$

when $h \in H_0^1$ (see [**9**, Chapter 8]). The area integral converges absolutely. Let P_0 be the orthogonal projection of L^2 onto H_0^2, and define a linear functional Ψ on L^2 by

$$\Psi(f) = \frac{2}{\pi}\int_{\mathbf{D}}\!\!\int\left[(P_0 f)'g + (P_0 f)\frac{\partial g}{\partial z}\right]\log\frac{1}{|z|}\,dx\,dy. \tag{2.10}$$

Lemma 2.11. $\sup_{\|f\|_2\leqslant 1}|\Psi(f)| \leqslant C(\mu_g(\mathbf{D})^{1/2} + \|\nu_g\|_*^{1/2}\nu_g(\mathbf{D})^{1/2})$.

Proof. By Schwarz's inequality,

$$|\Psi(f)| \leqslant \frac{2}{\pi}\left(\int_{\mathbf{D}}\!\!\int \frac{|(P_0 f)'|^2}{|z|}\log\frac{1}{|z|}\,dx\,dy\right)^{1/2}\left(\int_{\mathbf{D}}\!\!\int |g|^2|z|\log\frac{1}{|z|}\,dx\,dy\right)^{1/2}$$
$$+\frac{2}{\pi}\left(\int_{\mathbf{D}}\!\!\int \left|\frac{P_0 f}{z}\right|^2\left|\frac{\partial g}{\partial z}\right||z|\log\frac{1}{|z|}\,dx\,dy\right)^{1/2}\left(\int_{\mathbf{D}}\!\!\int \left|\frac{\partial g}{\partial z}\right||z|\log\frac{1}{|z|}\,dx\,dy\right)^{1/2}.$$

The four integrals are bounded respectively by C, $\mu_g(\mathbf{D})$, $C\|\nu_g\|_*$, and $\nu_g(\mathbf{D})$. □

Let $\mu \in L^2$ be the function on $\partial\mathbf{D}$ such that $\Psi(f) = \int fu\,d\theta/2\pi$ for all $f \in L^2$. By Lemma 2.11 we have

$$\|u\|_2^2 \leqslant C\big(\mu_g(\mathbf{D}) + \|\nu_g\|_*\nu_g(\mathbf{D})\big). \tag{2.12}$$

Lemma 2.13. (a) $u \in \overline{H_0^2}$.

(b) *There is a continuous function F on $\mathbf{D}$ such that F has L^1 boundary function u and $\partial f/\partial \bar z = g$.*

(c) *u and F are determined by* (a) *and* (b).

Proof. (a) follows from the fact that $\Psi(f) = 0$ when $f \in \overline{H^2}$. To prove (b), set

$$F(z) = \frac{-1}{2\pi i}\int_{\partial\mathbf{D}}\frac{u(\zeta)}{\bar\zeta - \bar z}\,d\bar\zeta + \frac{2}{\pi}\int_{\mathbf{D}}\!\!\int\log\left|\frac{1-\bar\zeta z}{\zeta - z}\right|\frac{\partial g}{\partial \zeta}\,d\xi\,d\eta.$$

F has L^1 boundary function u by (a) and well-known facts about Green's potential (e.g. [**21**, §IV. 10]). To prove $\partial F/\partial\bar{z} = g$ let V be any function with $\partial V/\partial\bar{z} = g$. $F - V$ is obviously harmonic, so

$$\frac{\partial(F-V)}{\partial\bar{z}}(w) = \lim_{r\to 1}\int \frac{e^{i\theta}}{(1-\bar{w}e^{i\theta})^2}\left(F(re^{i\theta}) - V(re^{i\theta})\right)\frac{d\theta}{2\pi}.$$

Since $e^{i\theta}/(1-\bar{w}e^{i\theta}) \in H_0^2$, (2.9) and (2.10) prove the limit is zero.

To show (c), assume (u_1, F_1) and (u_2, F_2) satisfy (a) and (b). Then $u_1 - u_2 \in \overline{H_0^2}$ and $u_1 - u_2$ is the boundary function of $F_1 - F_2$, which is analytic and therefore zero. □

We will call u the canonical boundary function for the equation $\partial V/\partial\bar{z} = g$. By Lemma 2.13 it is conformally invariant: if u is the canonical boundary function for $\partial V/\partial\bar{z} = g$ then $u \circ L_a - \int u\,dP_a$ is the canonical boundary function for $\partial V/\partial\bar{z} = g \circ L_a \cdot \overline{L_a'}$. It follows that

$$\left\|u \circ L_a - \int u\,dP_a\right\|_2^2 \leq C\left(\mu_{g\circ L_a\cdot\overline{L_a'}}(\mathbf{D}) + \|\nu_{g\circ L_a\cdot\overline{L_a'}}\|_*\nu_{g\circ L_a\cdot\overline{L_a'}}(\mathbf{D})\right)$$
$$\leq C(\Phi_1(a) + M\Phi_2(a)).$$

Equivalently, $\int |u - \int u\,dP_a|^2\,dP_a \leq C(\Phi_1(a) + M\Phi_2(a))$, proving Theorem 2.8. □

REMARK. The functions u and F may be obtained without using duality. Given g satisfying the conditions of Theorem 2.8, let

$$G(z) = \frac{2}{\pi}\log\left|\frac{1-\bar{\zeta}z}{\zeta - z}\right|\frac{\partial g}{\partial\bar{\zeta}}\,d\xi\,d\eta.$$

Then $g - \partial G/\partial\bar{z}$ is conjugate analytic; let $W = \int_0^{\bar{z}}(g - \partial G/\partial\bar{z})\,d\bar{z}$ be a primitive vanishing at the origin and take $F = W + G$ and $u(e^{i\theta}) = W(e^{i\theta})$.

THEOREM 2.14. *Suppose g is a function on $\mathbf{D}$ such that*

$$\int_{\mathbf{D}}\int |g\circ L_a|^2|L_a'|^2(1-|z|^2)\,dx\,dy \leq \Phi(a),$$

$$\int_{\mathbf{D}}\int \left|\frac{\partial g}{\partial z}\circ L_a\right||L_a'|^2(1-|z|^2)\,dx\,dy \leq \Phi(a),$$

where Φ is a bounded positive function on $\mathbf{D}$ with $\Phi(a) \to 0$ as $a \to \mathfrak{M}(B)$. Then there is a function F on $\mathbf{D}$ such that $\partial F/\partial\bar{z} = g$, having an L^1 boundary function $u \in Q_B$. Moreover, u satisfies estimates of the form $\|u\|_\infty \leq M$, $\int |u - \int u\,dP_a|^2\,dP_a \leq \Psi(a)$, where Ψ is a bounded function on $\mathbf{D}$ with $\Psi(a) \to 0$ as $z \to \mathfrak{M}(B)$, and M and Ψ depend only on Φ.

PROOF. Immediate from Theorems 2.7 and 2.8.

COROLLARY 2.15. *The Corona Theorem is true in QA_B: if $f_1,\ldots,f_n \in QA_B$ and* $\inf_z \max_j |f_j(z)| > 0$, *then there are $g_1,\ldots,g_n \in QA_B$ with $f_1g_1 + \cdots + f_ng_n \equiv 1$.*

PROOF. Mimic the proof in the appendix of [**16**] or Chapter 8 of [**9**]. □

Our next lemma is a technical result relating various shrinking conditions on measures. It is needed only for the proof of Corollaries 2.18 and 2.19.

LEMMA 2.16. *If μ is a positive Carleson measure on* $\mathbf{D}$ *then the following are equivalent.*

(i) $\int_{\mathbf{D}}(1-|a|^2)/|1-\bar{a}z|^2\, d\mu(z) \leq \Phi(a)$, *where* Φ *is bounded and* $\Phi(a) \to 0$ *as* $a \to \mathfrak{M}(B)$.

(ii) $\mu(S_a) \leq (1-|a|)\Psi(a)$, Ψ *bounded and* $\Psi(a) \to 0$ *as* $a \to \mathfrak{M}(B)$.

(iii) *For each* $\varepsilon > 0$ *there is a neighborhood* U *of* $\mathfrak{M}(B)$ *such that* $\|\chi_U \mu\|_* < \varepsilon$, *where* χ_U *is the characteristic function of* U.

REMARK. (1) If $d\mu = |g|(1-|z|^2)\,dx\,dy$ for some function g then (i) is equivalent by the change of variables formula to

$$\int_{\mathbf{D}} |g \circ L_a|\,|L_a'|^2\left(1-|z|^2\right) dx\,dy \leq \Phi(a).$$

(2) When $d\mu = |\nabla f|^2(1-|z|^2)\,dx\,dy$ with f harmonic, then (i)–(iii) are all necessary and sufficient for $f \in \mathrm{VMO}_B$ [3]. That the conditions are equivalent in general is undoubtedly known to many people, but there is no proof in print. We thank D. Marshall for the proof that (ii) implies (i).

PROOF OF LEMMA 2.16. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The implication from (i) to (ii) follows from the inequality $1/(1-|a|) \leq C(1-|a|^2)/|1-\bar{a}z|^2$ for $z \in S_a$, while the proof that (ii) $\Rightarrow$ (iii) given in [3, Lemma 5] for the case $\mu = |\nabla f|^2(1-|z|^2)\,dx\,dy$ goes over verbatim.

(iii) $\Rightarrow$ (ii): Use induction to choose Blaschke products $b_j \in B^{-1}$, $b_j \mid b_{j+1}$, the numbers δ_j, $0 < \delta_j < 1$, such that the following statements are true. For each $n \geq 1$, $(\delta_n, 1/2^n, \delta_{n+1})$ plays the role of $(\eta, \varepsilon, \kappa)$ in Lemma 1.1. $\|\chi_{G_n}\mu\|_* \to 0$ as $n \to \infty$, where $G_n = \{z\colon |b_n(z)| > \delta_n\}$. If $a \in \mathbf{D}$, let $n = n(a)$ be an index such that $|b_{n+1}(a)| > \delta_{n+1}$; we can make $n(a)$ go to ∞ as $a \to \mathfrak{M}(B)$. We have then

$$\begin{aligned}\mu(S_a) &= \mu(S_a \cap G_n) + \mu(S_a \backslash G_n)\\ &\leq \mu(\{z \in S_a\colon |b_n(z)| > \delta_n\}) + \mu(\{z \in S_a\colon |b_{n+1}(z)| \leq \delta_n\})\\ &\leq (1-|a|)\left(\|\chi_{G_n}\mu\|_* + \frac{1}{2^n}\|\mu\|_*\right)\\ &= o(1-|a|) \quad \text{as } a \to \mathfrak{M}(B).\end{aligned}$$

(ii) $\Rightarrow$ (i): Fix $a \in \mathbf{D}$ and define a sequence $\{z_j\}$ by $z_j = a(1-1/2^j)/|a|$. Let N satisfy $|z_N| < |a| \leq a_{n+1}$; write

$$\int_{\mathbf{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2}\, d\mu(z) = \int_{S_{z_N}} \frac{1-|a|^2}{|1-\bar{a}z|^2}\, d\mu(z) + \sum_{j=0}^{N-1} \int_{S_{z_j}} \frac{1-|a|^2}{|1-\bar{a}z|^2}\, d\mu(z).$$

In these integrals,

$$\frac{1-|a|^2}{|1-\bar{a}z|^2} \leq C2^N \quad \text{for } z \in \mathbf{D}$$

and

$$\frac{1-|a|^2}{|1-\bar{a}z|^2} \leq C \cdot 2^{2j}2^{-N}$$

when $z \in \mathbf{D}\setminus S_{z_j}$. So

$$\int_{\mathbf{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2}\,d\mu(z) \leq C2^{-N}\left(2^N\Psi(z_N) + \sum_{j=0}^{N-1} 2^{2j}2^{-N}\Psi(z_j)\right),$$

proving

$$\int_{\mathbf{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2}\,d\mu(z) \leq C2^{-N}\sum_{j=0}^{N} 2^j\Psi(z_j). \tag{2.17}$$

If $0<\eta<1$ and M is a positive integer, then by Schwarz's Lemma there is $0<\delta<1$ such that whenever $|z_N|<|a|\leq|z_{N+1}|$ and b is a Blaschke product with $|b(a)|>\delta$ we have $|b(z_j)|>\eta$ for $N-M\leq j\leq N$. To prove (i), fix $\varepsilon>0$. We need $0<\delta<1$ such that $|b(a)|>\delta$ implies $\int_{\mathbf{D}}(1-|a|^2)/|1-\bar{a}z|^2\,d\mu(z)<\varepsilon$. Let M be some integer with $2^{-M}<\varepsilon[C(2+\sup_{w\in\mathbf{D}}\Psi(w))]^{-1}$ with C as in (2.17). Find a Blaschke product $b\in B^{-1}$ and $0<\eta<1$ such that $|b(z)|>\eta$ implies $\Psi(z)<\varepsilon[C(2+\sup_{w\in\mathbf{D}}\Psi(w))]^{-1}$. Use the remark following (2.17) to choose $0<\delta<1$ corresponding to η, M. Then if $|b(a)|>\delta$, (2.17) gives

$$\begin{aligned}\int_{\mathbf{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2}\,d\mu(z) &\leq C2^{-N}\left(\sum_{j=(N-M)^+}^{N} 2^j\Psi(z_j) + \sum_{j=0}^{(N-M)^+-1} 2^j\Psi(z_j)\right)\\ &\leq C2^{-N}\left(\varepsilon 2^{N+1}\left[C\left(2+\sup_{w\in\mathbf{D}}\Psi(w)\right)\right]^{-1} + 2^{N-M}\sup_{w\in\mathbf{D}}\Psi(w)\right)\\ &<\varepsilon. \quad \square\end{aligned}$$

Definition. We will call a measure μ satisfying (i)–(iii) of Lemma 2.16 a *B-Carleson measure*.

Corollary 2.18. *If μ is a B-Carleson measure then there is a nonnegative function τ on $\mathbf{D}$ such that $\tau(z)\to\infty$ as $z\to\mathfrak{M}(B)$ and $\tau\mu$ is still B-Carleson.*

Proof. Choose a sequence $\{b_n\}$ of Blaschke products in B^{-1} such that $b_n\,|\,b_{n+1}$ and an increasing sequence of positive numbers $\{\delta_n\}$ such that $\delta_n\to 1$ as $n\to\infty$ and $\sum_{n=1}^{\infty}\|\chi_{G_n}\mu\|_*<\infty$, where $G_n=\{z\in\mathbf{D}\colon |b_n(z)|>\delta_n\}$. Then $G_n\supseteq G_{n+1}$ and $\bigcap_n G_n=\varnothing$. Let $\{t_n\}$ be a sequence of positive numbers such that $t_n\to\infty$ as $n\to\infty$ and $\sum_{n=1}^{\infty}t_n\|\chi_{G_n}\mu\|_*<\infty$. Define $\tau(z)=1$ if $z\notin G_1$ and $\tau(z)=t_n$ if $z\in G_n\setminus G_{n+1}$, $n\geq 1$. Then $\tau\mu$ is Carleson and $\|\chi_{G_n}\tau\mu\|_*\to 0$ as $n\to\infty$, hence (iii) of Lemma 2.16 is satisfied. $\square$

Corollary 2.19. *Suppose $\{z_n\}$ is an interpolating sequence for H^∞, $0<\eta<1$, μ_j is a measure supported on $\{z\in\mathbf{D}\colon \rho(z,z_j)<\eta\}$, and the total variation of μ_j, $|\mu_j|(\mathbf{D})$, is bounded as j varies and tends to zero as $z_j\to\mathfrak{M}(B)$. Then $(1-|z|)\Sigma\mu_j$ is a B-Carleson measure.*

Proof. This follows immediately from (iii) of Lemma 2.16 and the fact (see [**9**, Chapter 7]) that $\Sigma\delta_{z_j}(1-|z_j|)$ is a Carleson measure. $\square$

3. Turning approximate interpolation into actual interpolation. In this section we prove that if $\{z_n\}$ is an H^∞ interpolating sequence satisfying an auxiliary condition Λ_B, then any bounded sequence of numbers that can be approximately interpolated by a Q_B function at $\{z_n\}$ can be interpolated by a QA_B function. The condition Λ_B is a technical condition needed for the construction of certain analytic functions which multiply the Blaschke product with zeros $\{z_n\}$ into QA_B. We will show in §4 that any sequence which is thin near $\mathfrak{M}(B)$ satisfies Λ_B.

DEFINITION. Let B be a Douglas algebra. A sequence $\{z_n\} \subseteq \mathbf{D}$ is said to satisfy Λ_B if the following holds: whenever σ is a function on $\mathbf{D}$ such that $\sigma \geqslant 4$ and $\sigma(z) \to \infty$ as $z \to \mathfrak{M}(B)$, there is $v \in \mathrm{VMO}_B$ with $0 \leqslant v(z) \leqslant \sigma(z)$ for all $z \in \mathbf{D}$ and $v(z_n) \to \infty$ as $z_n \to \mathfrak{M}(B)$.

THEOREM 3.1. *Suppose B is a Douglas algebra and $\{z_n\}$ is an H^∞ interpolating sequence satisfying Λ_B. Then if $\{\lambda_n\}$ is a bounded sequence of complex numbers and $u \in Q_B$ with $|u(z_n) - \lambda_n| \to 0$ as $z_n \to \mathfrak{M}(B)$, there is $h \in QA_B$ with $h(z_n) = \lambda_n$ for all n.*

REMARK. We mentioned in the Introduction that we will need specific estimates to carry out the linearization argument in §6. Because of the estimates in §2 it will be clear from the proof that the following version of Theorem 3.1 is in fact true. Suppose $\{z_n\}$ satisfies Λ_B. Suppose $\varepsilon_n > 0$ is a bounded sequence with $\varepsilon_n \to 0$ as $z_n \to \mathfrak{M}(B)$ and Φ is a bounded nonnegative function on $\mathbf{D}$ such that $\Phi(z) \to 0$ as $z \to \mathfrak{M}(B)$. Then there exist $C > 0$ and a bounded nonnegative function Ψ on $\mathbf{D}$ with $\Psi(z) \to 0$ as $z \to \mathfrak{M}(B)$, such that if $\|u\|_\infty \leqslant 1$, $M_I(u) \leqslant \Psi(z_I)$ for all arcs I, and $|u(z_n) - \lambda_n| < \varepsilon_n$ for all n, then there is $h \in H^\infty$ with $h(z_n) = \lambda_n$ for all n and $\|h\|_\infty \leqslant C$, $M_I(h) \leqslant \Phi(z_I)$ for all arcs I.

PROOF OF THEOREM 3.1. The first step is to modify u slightly to obtain a function constant on small hyperbolic discs around the points z_n. Let $\{z_n\}$, u, $\{\lambda_n\}$ be as in the hypothesis and let $\eta > 0$ be some number small enough so that the discs $\{z \in \mathbf{D}: |z - z_n| < 4\eta(1 - |z_n|)\}$ are disjoint and contained in $\mathbf{D}$. Let γ be a smooth function on $[0, \infty)$ such that $0 \leqslant \gamma \leqslant 1$ and $\gamma(t) = 0$ when $t \leqslant 1$, $\gamma(t) = 1$ when $t \geqslant 2$. Define a function g by

$$g(z) = \begin{cases} \gamma\left(\dfrac{|z - z_n|}{\eta(1 - |z_n|)}\right)(u(z_n) - \lambda_n) + \lambda_n & \text{if } |z - z_n| \leqslant 2\eta(1 - |z_n|), \\ u(z) \quad \text{if } |z - z_n| > 2\eta(1 - |z_n|) \text{ for all } n. \end{cases}$$

Let $\varepsilon_n = \max\{|u(z) - \lambda_n| + (1 - |z_n|)|\nabla u(z)| : |z - z_n| \leqslant 2\eta(1 - |z_n|)\}$. Since $u \in \mathrm{VMO}_B$ and $|u(z_n) - \lambda_n| \to 0$ as $z_n \to \mathfrak{M}(B)$, easy computations show that the ε_n are bounded and tend to zero as $z_n \to \mathfrak{M}(B)$, and that

$$|\nabla g(z)| \leqslant C\varepsilon_n/(1 - |z_n|), \quad |\Delta g(z)| \leqslant C\varepsilon_n/(1 - |z_n|)^2,$$

with C independent of n, whenever $|z - z_n| \leqslant 2\eta(1 - |z_n|)$. These facts together with Theorem 2.1 and Corollary 2.19 show that $|\nabla g|^2(1 - |z|^2)\,dx\,dy$ and $|\Delta g|(1 - |z|^2)\,dx\,dy$ are B-Carleson measures. Also note that $g(z_n) = \lambda_n$ when $|z - z_n| \leqslant \eta(1 - |z_n|)$.

Next use Corollary 2.18 to choose a function σ on $\mathbf{D}$ such that $\sigma \geqslant 4$ and $\sigma(z) \to \infty$ as $z \to \mathfrak{M}(B)$, such that $e^{2\sigma(z)}|\nabla g(z)|^2(1 - |z|^2)\,dx\,dy$ and $e^{\sigma(z)}|\Delta g(z)|(1 - |z|^2)\,dx\,dy$ are still B-Carleson measures. By condition Λ_B there is $v \in \mathrm{VMO}_B$ such that $0 \leqslant v(z) \leqslant \sigma(z)$ for all $z \in \mathbf{D}$ and $v(z_n) \to \infty$ as $z_n \to \mathfrak{M}(B)$. Let b be the Blaschke product with zeros $\{z_n\}$ and consider the $\bar{\partial}$-equation

$$\frac{\partial}{\partial \bar{z}}\left(g + be^{-(v+i\tilde{v})}\alpha\right) = 0 \tag{3.2}$$

for the unknown function α.

LEMMA 3.3. *If b is an interpolating Blaschke product with zeros $\{z_n\}$, $\psi \in QA_B$ has no zeros, and $\psi(z_n) \to 0$ as $z_n \to \mathfrak{M}(B)$, then $\psi b \in QA_B$.*

PROOF. We may assume $\|\psi\|_\infty \leqslant 1$. We first show that if $0 < \rho < 1$ and $\varepsilon > 0$ are given we can find a neighborhood V of $\mathfrak{M}(B)$ such that $z \in V$ and $|b(z)| < 1 - \rho$ imply $|\psi(z)| < \varepsilon$. Let M be a number such that $\sup_n |\lambda_n| \leqslant 1$ implies there is $h \in H^\infty$ with $h(z_n) = \lambda_n$ for all n and $\|h\|_\infty \leqslant M$. We may assume ε is so small that

$$M\varepsilon + (1 + M\varepsilon)\frac{(1-\rho)}{(1-\frac{1}{2}\rho)} < 1.$$

Choose N so that

$$\left[M\varepsilon + (1 + M\varepsilon)\frac{(1-\rho)}{(1-\frac{1}{2}\rho)}\right]^N < \varepsilon.$$

Let U be a neighborhood of $\mathfrak{M}(B)$ such that $z_n \in U$ implies that $|\psi(z_n)| < \varepsilon^N$, and let b_U be the Blaschke product with zeros $\{z_n\colon z_n \in U\}$. Then $b/b_U \in B^{-1}$, hence $V = \{z \in U\colon |(b/b_U)(z)| > 1 - \frac{1}{2}\rho\}$ is a neighborhood of $\mathfrak{M}(B)$. Now say $z \in V$ is such that $|b(z)| < 1 - \rho$. Then $|b_U(z)| < (1-\rho)/(1-\frac{1}{2}\rho)$. Let $f \in H^\infty$, $\|f\| \leqslant M\varepsilon$ be such that $f(z_n) = \psi(z_n)^{1/N}$ for $z_n \in U$. Then b_U divides $f - \psi^{1/N}$, hence

$$|f(z) - \psi(z)^{1/N}| = |((f - \psi^{1/N})/b_U)(z)|\,|b_U(z)| < (M\varepsilon + 1)(1-\rho)/(1-\tfrac{1}{2}\rho).$$

This implies $|\psi(z)|^{1/N} < M\varepsilon + (M\varepsilon + 1)(1-\rho)/(1-\frac{1}{2}\rho)$, so $|\psi(z)| < \varepsilon$.

Now let $\varepsilon > 0$ be given. Write

$$\begin{aligned}\int |\psi b - (\psi b)(z)|^2\,dP_z &= \int |\psi|^2\,dP_z - |\psi(z)|^2|b(z)|^2 \\ &= \int |\psi|^2\,dP_z - |\psi(z)|^2 + |\psi(z)|^2\left(1 - |b(z)|^2\right) \\ &\leqslant \int |\psi - \psi(z)|^2\,dP_z + \max\left(|\psi(z)|^2, 1 - |b(z)|^2\right).\end{aligned}$$

Since $\psi \in QA_B$, we are done by Theorem 2.1. □

Now equation (3.2) is equivalent to

$$\frac{\partial \alpha}{\partial \bar{z}} = -\frac{e^{v+i\tilde{v}}}{b}\frac{\partial g}{\partial \bar{z}}. \tag{3.4}$$

We claim there is a solution α of (3.4) with L^1 boundary function in Q_B. Supposing this to be true, Theorem 3.1 follows by letting h be $g + be^{-(v+i\tilde{v})}\alpha$; then $\partial h/\partial \bar{z} = 0$, $h(z_n) = \lambda_n$ for all n, and $h \in QA_B$ since g, α, and $e^{-(v+i\tilde{v})}b$ (by Lemma 3.3) all have boundary functions in Q_B.

To prove the claim let $Q=(e^{v+i\tilde{v}}/b)(\partial g/\partial\bar{z})$. Theorem 2.14 will give α provided $|Q|^2(1-|z|^2)\,dx\,dy$ and $|\partial Q/\partial z|(1-|z|^2)\,dx\,dy$ are B-Carleson measures. We only show that $|\partial Q/\partial z|(1-|z|^2)\,dx\,dy$ is B-Carleson; the proof for $|Q|^2(1-|z|^2)\,dx\,dy$ is simpler. First of all,

$$\frac{\partial Q}{\partial z}=\frac{e^{v+i\tilde{v}}}{4b}\Delta g+2\frac{e^{v+i\tilde{v}}}{b}\frac{\partial v}{\partial z}\frac{\partial g}{\partial\bar{z}}-\frac{e^{v+i\tilde{v}}}{b^2}\frac{\partial b}{\partial z}\frac{\partial g}{\partial\bar{z}}. \tag{3.5}$$

Since b is interpolating and g is constant on small hyperbolic discs around the points $\{z_n\}$, $|b|$ has a positive lower bound on the set where $\partial Q/\partial z\neq 0$ [**12**, Lemma 4.2]. By definition, $|e^{v+i\tilde{v}}|\leq e^{\sigma}$. Applying Schwarz's inequality to the last two terms in (3.5), we obtain

$$\begin{aligned}
&\int_{\mathbf{D}}\!\!\int\left|\frac{\partial Q}{\partial z}\right|(1-|z|^2)\frac{1-|a|^2}{|1-\bar{a}z|^2}\,dx\,dy\\
&\quad\leq C\Bigg[\int_{\mathbf{D}}\!\!\int e^{\sigma(z)}|\Delta g(z)|\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}\,dx\,dy\\
&\qquad+\left(\int_{\mathbf{D}}\!\!\int e^{2\sigma(z)}\left|\frac{\partial g}{\partial\bar{z}}\right|^2\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}\,dx\,dy\right)^{1/2}\\
&\qquad\times\left(\int_{\mathbf{D}}\!\!\int\left|\frac{\partial v}{\partial z}\right|^2\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}\,dx\,dy\right)^{1/2}\\
&\qquad+\left(\int_{\mathbf{D}}\!\!\int e^{2\sigma(z)}\left|\frac{\partial g}{\partial\bar{z}}\right|^2\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}\right)^{1/2}\\
&\qquad\qquad\times\left(\int_{\mathbf{D}}\!\!\int\left|\frac{\partial b}{\partial z}\right|^2\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}\,dx\,dy\right)^{1/2}\Bigg]\\
&\quad=C[\mathrm{I}+\mathrm{II}^{1/2}\mathrm{III}^{1/2}+\mathrm{IV}^{1/2}\mathrm{V}^{1/2}].
\end{aligned}$$

We know that $e^{\sigma}|\Delta g|(1-|z|^2)\,dx\,dy$ and $e^{2\sigma}|\nabla g|^2(1-|z|^2)\,dx\,dy$ satisfy (i) of Lemma 2.16; let $\Phi(a)$ be the function given there. Then the integrals I, II, III, IV, V are bounded by $\Phi(a)$, $\Phi(a)$, $\|v\|_*^2$, $\Phi(a)$, 1, respectively; since $\Phi(a)+\Phi(a)^{1/2}\|v\|_*+\Phi(a)^{1/2}\to 0$ as $a\to\mathfrak{M}(B)$, Theorem 3.1 is proved. □

We now give an example showing that Theorem 3.1 becomes false if the condition Λ_B is not assumed to hold. We will work in the upper half plane. Let BUC denote the algebra of bounded uniformly continuous functions on $\mathbf{R}$. Then [**18**] $H^\infty+\mathrm{BUC}$ is a Douglas algebra and $Q_{H^\infty+\mathrm{BUC}}=\mathrm{VMO}_{\mathbf{R}}\cap L^\infty$, $QA_{H^\infty+\mathrm{BUC}}=\mathrm{VMO}_{\mathbf{R}}\cap H^\infty$. Here $\mathrm{VMO}_{\mathbf{R}}=\{f\in L^1_{\mathrm{loc}}(\mathbf{R})\colon (1/|I|)\int_I|f-I(f)|\,dx\to 0 \text{ as } |I|\to 0\}$, where I runs through all intervals of $\mathbf{R}$.

EXAMPLE. There is an interpolating sequence $\{w_n\}$ for H^∞ and a function $u\in\mathrm{BUC}$ such that no $v\in\mathrm{VMO}_{\mathbf{R}}\cap H^\infty$ satisfies $v(w_n)=u(w_n)$ for all n.

Construction of the Example. For each positive integer k let x_k and t_k be positive numbers satisfying $x_k+t_k<x_{k+1}$ and $t_k\to\infty$ as $k\to\infty$. Let $\{z_j^{(k)}\}$ be the finite

sequence $\{x_k + j/2^k + i/2^k\}_{0<j<2^k t_k}$. Let $\{w_n\} = \bigcup_{k=1}^{\infty}\{z_j^{(k)}\}$. Then $\{w_n\}$ is an interpolating sequence for H^∞. Let b be the Blaschke product with zeros $\{i+j$: $j \in \mathbf{Z}\}$, and let $u = \bar{b} \in \mathrm{BUC}$. Suppose there were $v \in \mathrm{VMO}_{\mathbf{R}} \cap H^\infty$ satisfying $v(w_n) = u(w_n)$ for all n. The function $f = vb$ would satisfy $f \in \mathrm{VMO}_{\mathbf{R}} \cap H^\infty$, $f(w_n) \to 1$ as $n \to \infty$, and $f(i+j) = 0$ for all $j \in \mathbf{Z}$. The first two of these would show that for any $\varepsilon > 0$,

$$\sup\left\{\left|(N, N+1) \cap \{x\colon |f(x) - 1| > \varepsilon\}\right|\colon N \in (x_k, x_k + t_k - 1)\right\} \to 0 \quad \text{as } k \to \infty.$$

This would imply by Poisson's formula that

$$\varliminf_{n\to\infty} |f(i+n) - 1| = 0, \quad \text{a contradiction.} \quad \square$$

There are Douglas algebras B (e.g. $H^\infty + C$ and $H^\infty + L^\infty_{\partial\mathbf{D}\setminus\{1\}}$) such that any H^∞ interpolating sequence satisfies Λ_B. Certain questions which are open for general Douglas algebras can be answered for these algebras using Theorem 3.1; for example, the analogue of Theorem 2 of **[22]** is true.

4. A construction. In this section we show that if the interpolating sequence $\{z_n\}$ is thin near $\mathfrak{M}_B$, then it satisfies Λ_B. Our construction is based on that of Garnett and Jones **[10]**.

DEFINITION. Let I be an arc and f be a Lipschitz function on $\partial\mathbf{D}$. We say f is *(a, b)-adapted to I* if f is supported in $\tilde{I}$, $|f| \leq a$, and $|df/d\theta| \leq b/|I|$.

LEMMA 4.1 **[10]**. *Suppose $\mathcal{G} = \{I_j\}$ is a sequence of arcs satisfying $\sum_{I_j \subseteq J} |I_j| \leq M|J|$ for all arcs J. Let a_j be (a, b)-adapted to I_j, and let $f = \sum_j a_j$. Then $f \in \mathrm{BMO}$ and $\|f\|_* \leq CM(a+b)$.* $\square$

Garnett and Jones state Lemma 4.1 for dyadic I_j, but there is no difficulty in modifying their proof to cover the nondyadic case. We give two variations on Lemma 4.1.

DEFINITION. Let $\mathcal{G}$ be a family of arcs and J an arc. The *density of $\mathcal{G}$ in J*, $D_{\mathcal{G}}(J)$, is $|\bigcup_{I \in \mathcal{G};\, I \subseteq J} I|/|J|$.

LEMMA 4.2. *With $\mathcal{G} = \{I_j\}, \{a_j\}, f$, and (a, b) as in Lemma 4.1, suppose L is an arc for which $L \cap \tilde{I}_j \neq \varnothing$ implies $|I_j| \leq |L|$. Then $M_L(f) \leq 2L(|f|) \leq CaMD_{\mathcal{G}}(\tilde{\tilde{L}})$.*

PROOF. The first inequality is trivial. The function a_j vanishes identically on L if $L \cap \tilde{I}_j = \varnothing$. If $L \cap \tilde{I}_j \neq \varnothing$ then $I_j \subseteq \tilde{\tilde{L}}$. Let $E = \bigcup_{I_j \subseteq \tilde{\tilde{L}}} I_j$, and let $E = \bigcup E_n$ be the decomposition of E into disjoint open arcs. Then

$$L(|f|) \leq \frac{a}{|L|} \sum_{I_j \subseteq \tilde{\tilde{L}}} |I_j| = \frac{a}{|L|} \sum_n \sum_{I_j \subseteq E_n} |I_j| \leq \frac{aM}{|L|} \sum_n |E_n| \leq aMD_{\mathcal{G}}(\tilde{\tilde{L}}). \quad \square$$

LEMMA 4.3. *With $\mathcal{G} = \{I_j\}, \{a_j\}, f, (a, b)$ as in Lemma 4.1, let L be an arc and suppose that $L \cap \tilde{I}_j \neq \varnothing$ implies $|I_j| \geq |L|$. Let $\gamma = \sup_{L \subseteq \tilde{\tilde{I}}_j} |L|/|I_j|$. Then $M_L(f) \leq V_L(f) \leq CMb\gamma$.*

PROOF. Again the first inequality is trivial. If a_j does not vanish identically on L, then $L \subseteq \tilde{\tilde{I}}_j$. For $n \geqslant 1$, let

$$\mathcal{G}_n = \left\{ I_j \colon L \subseteq \tilde{\tilde{I}}_j \text{ and } 3^{n-1} \leqslant |I_j|/|L| < 3^n \right\},$$

and let d_n be the cardinality of $\mathcal{G}_n$. Then $d_n = 0$ if $3^{-n} \geqslant \gamma$. For any n we have $\bigcup_{I_j \in \mathcal{G}_n} I_j \subseteq 10 \cdot 3^n L$. Therefore

$$d_n |L| \cdot 3^{n-1} \leqslant \sum_{I_j \in \mathcal{G}_n} |I_j| \leqslant C10 \cdot 3^n |L|, \quad \text{or} \quad d_n \leqslant 30C.$$

If $I_j \in \mathcal{G}_n$, then the variation of a_j on L is at most $b/3^{n-1}$. So

$$V_L(f) \leqslant \sum_{\{n \colon 3^{-n} < \gamma\}} 30Cb/3^{n-1} \leqslant 45Mb\gamma. \quad \square$$

COROLLARY 4.4. *With $\{I_j\}$ and $\{a_j\}$ as in Lemma 4.1, suppose there is a Blaschke product $b \in B^{-1}$ and a number $0 < \eta < 1$ such that $|b(z)| < \eta$ when $z \in \bigcup_j T_{I_j}$. Then $f \in \mathrm{VMO}_B$.*

PROOF. Given $\varepsilon > 0$ we will find $0 < \rho < 1$ such that $|b(z_L)| > \rho$ implies $M_L(f) < C\varepsilon$. Make ρ large enough so that $|b(z_L)| > \rho$ implies

(4.5) $$D_{\mathcal{G}}(L) \leqslant \varepsilon/aM.$$

(4.6) $$\text{If } |I_j| \geqslant |L| \text{ and } L \subseteq \tilde{\tilde{I}}_j, \text{ then } |L|/|I_j| < \varepsilon/bM.$$

In fact, (4.5) follows from Lemma 1.1 and (4.6) from Schwarz's Lemma, provided ρ is sufficiently large. Now use Lemmas 4.2 and 4.3. $\square$

We now give the main construction of **[10]** in the form we will use.

THEOREM 4.7 (GARNETT-JONES **[10]**). *Let $N < \infty$ and $\varepsilon > 0$ be given. Then there is $\mu > 0$ such that the following will be true.*

Let $\mathfrak{R}$ and $\mathfrak{B}$ be two collections of distinct dyadic arcs satisfying $\min(D_{\mathfrak{R}}(L), D_{\mathfrak{B}}(L)) < \mu$ for all arcs L. Then there exist collections of distinct dyadic arcs $G(\mathfrak{R}) \supseteq \mathfrak{R}$ and $G(\mathfrak{B}) \supseteq \mathfrak{B}$, with $G(\mathfrak{R}) \cap G(\mathfrak{B}) = \varnothing$, and functions a_j which are $(2, 160 \cdot 3^3)$-adapted to arcs $I_j \in G(\mathfrak{R}) \cup G(\mathfrak{B})$, such that the following will hold with $f = \Sigma a_j$:

(i) $\Sigma_{I_j \in G(\mathfrak{R}) \cup G(\mathfrak{B}), I_j \subseteq I} |I_j| \leqslant 3|I|$ *for all arcs I, $I_j \subseteq I$.*

(ii) $0 \leqslant f \leqslant N$.

(iii) $I(f) > N - \varepsilon$ *if $I \in \mathfrak{R}$.*

(iv) $I(f) < \varepsilon$ *if $I \in \mathfrak{B}$.*

(v) *There is $d > 0$ such that $D_{\mathfrak{R} \cup \mathfrak{B}}(I_j) \geqslant d$ for all $I_j \in G(\mathfrak{R}) \cup G(\mathfrak{B})$.*

In particular, $\|f\|_* \leqslant C$ *by Lemma* 3.1. $\square$

REMARKS. (1) Theorem 4.7 is proved, though not stated explicitly, in **[10]**. Statement (v) follows from the fact that only a fixed finite number of generations is used in the construction in **[10]**. See **[14]** for further discussion.

(2) The function f belongs to VMO_B if the following additional condition is satisfied: there exist a Blaschke product $b \in B^{-1}$ and a number $0 < \eta < 1$ such that $I \in \mathfrak{R} \cup \mathfrak{B}$ implies $|b(z)| < \eta$ whenever $z \in T_I$. To see this, note that by (v) and Lemma 1.1 there is $0 < \eta' < 1$ such that $J \in G(\mathfrak{R}) \cup G(\mathfrak{B})$ implies $|b(z)| < \eta'$ for all $z \in T_J$, and use Corollary 4.4.

Another important step in our construction is the following lemma.

LEMMA 4.8. *Let* $\mu > 0$ *and* $0 < \eta < 1$ *be given. Then there are numbers* $0 < \delta < 1$ *and* $0 < \theta < 1$ *making the following statements true.*

Let $f \in H^\infty$ *with* $\|f\|_\infty = 1$ *and let* $\{z_n\}$ *be a Blaschke sequence. Suppose*

(i) $\prod_{m \neq n} \rho(z_m, z_n) > \theta$ *for all* n,

(ii) $|f(z_n)| > \delta$ *for all* n.

Let $\mathfrak{R} = \{I: I$ *is a dyadic arc and* $\exists z_n \in T_I\}$, $\mathfrak{B} = \{I: I$ *is a dyadic arc and* $\exists z \in T_I, |f(z)| < \eta\}$.

Then $\min(D_{\mathfrak{R}}(J), D_{\mathfrak{B}}(J)) < \mu$ *for all arcs* J. □

PROOF. If ϕ is any H^∞ function define $\mathcal{E}_{\phi,\eta} = \{I: I$ is a dyadic arc and $\exists z \in T_I, |\phi(z)| \leq \eta\}$. By Lemma 1.1 there is $0 < \rho < 1$ such that if $\|\phi\|_\infty \leq 1$ and $|\phi(z_j)| \geq \rho$, then $D_{\mathcal{E}_{\phi,\eta}}(J) < \mu$. Let b be the Blaschke product with zeros $\{z_n\}$. Using Lemma 4.2 of **[12]**, choose θ large enough so that there exists $0 < \omega < 1$ satisfying: (i) and $|b(z)| \leq \rho$ force $\inf_n |(z - z_n)/(1 - \bar{z}_n z)| \leq \omega$. Then use Schwarz's Lemma to choose δ large enough so that (ii) and $|f(z)| \leq \rho$ force $\inf_n |(z - z_n)/(1 - \bar{z}_n z)| > \omega$.

Let J be any arc. If $\inf_n |(z_J - z_n)/(1 - \bar{z}_J z_n)| \leq \omega$ then $|f(z_J)| > \rho$; if $\inf_n |(z_J - z_n)/(1 - \bar{z}_J z_n)| > \omega$ then $|b(z_J)| > \rho$. In either case

$$\min(D_{\mathfrak{R}}(J), D_{\mathfrak{B}}(J)) \leq \min\left(D_{\mathcal{E}_{b,\eta}}(J), D_{\mathcal{E}_{f,\eta}}(J)\right) < \mu. \quad \square$$

We now give the main result of this section.

THEOREM 4.9. *Suppose* $\{z_n\}$ *is thin near* $\mathfrak{M}(B)$. *If* σ *is a function on* $\mathbf{D}$ *satisfying* $\sigma \geq 4$ *and* $\sigma(z) \to \infty$ *as* $z \to \mathfrak{M}(B)$, *then there is* $u \in \mathrm{VMO}_B$ *such that* $0 \leq u(z) \leq \sigma(z)$ *for all* $z \in \mathbf{D}$ *and* $u(z_n) \to \infty$ *as* $z_n \to \mathfrak{M}(B)$.

PROOF. By Schwarz's Lemma and the BMO analogue of Corollary 2.4, which can be established by a similar proof, there is a number $\lambda > 0$ with the property that if $f \in \mathrm{BMO}$ satisfies $\|f\|_* \leq \lambda$, then $|f(z) - I_z(f)| \leq 1$ for any arc I and any $z \in T_I$. Let $\{\varepsilon_j\}$ be a sequence of positive numbers such that $\Sigma\varepsilon_j \leq \lambda$. We will construct u as a sum of VMO_B functions u_j, each of which satisfies $\|u_j\|_* < \varepsilon_j$.

By Theorem 4.7 there are $\mu_j > 0$ such that the following statements will hold. Let $\mathfrak{R}_j$ and $\mathfrak{B}_j$ be collections of distinct dyadic arcs satisfying $\min(D_{\mathfrak{R}_j}(L), D_{\mathfrak{B}_j}(L)) < \mu_j$ for all arcs L. Then there exist families of distinct dyadic arcs $G(\mathfrak{R}_j) \supseteq \mathfrak{R}_j$, $G(\mathfrak{B}_j) \supseteq \mathfrak{B}_j$, and a function u_j such that

(4.10) $$\|u_j\|_* < \varepsilon_j \quad \text{and} \quad 0 \leq u_j \leq j,$$

(4.11) $$I(u_j) > j - 1/2^j \quad \text{if } I \in \mathfrak{R}_j,$$

(4.12) $$I(u_j) < 1/2^j \quad \text{if } I \in \mathfrak{B}_j,$$

(4.13) u_j is a sum of functions $(a^{(j)}, b^{(j)})$-adapted to arcs in $G(\mathfrak{R}_j) \cup G(\mathfrak{B}_j)$,

(4.14) there is $d_j > 0$ such that $D_{\mathfrak{R}_j \cup \mathfrak{B}_j}(I) \geq d_j$ for all $I \in G(\mathfrak{R}_j) \cup G(\mathfrak{B}_j)$.

We will inductively choose Blaschke products $b_j \in B^{-1}$ such that $b_j \mid b_{j+1}$, and numbers $0 < \eta_j < 1$ with $\eta_{j+1} > \eta_j$, such that the following will hold.

For $j \geqslant 1$, if

(4.15)
$$\mathcal{R}_j^0 = \{I\colon I \text{ is a dyadic arc and } \exists z_n \in T_I, |b_j(z_n)| > \eta_j\},$$
$$\mathcal{B}_j^0 = \{I\colon I \text{ is a dyadic arc and } \exists z \in T_I, |b_{j-1}(z)| \leqslant \eta_{j-1}\},$$
then $\min\left(D_{\mathcal{R}_j^0}(L), D_{\mathcal{B}_j^0}(L)\right) \leqslant \mu_j$ for all arcs L.

(4.16)
$$|b_j(z)| \geqslant \eta_j \quad \text{implies } \sigma(z) > 4 + j^2.$$

Start by setting $b_0 = 1$, $\eta_0 = 0$. Now let $j \geqslant 1$ and suppose b_{j-1} and η_{j-1} have been chosen. Let δ and θ be such that $(\eta_{j-1}, \mu_j, \delta, \theta)$ plays the role of $(\eta, \mu, \delta, \theta)$ in Lemma 4.8.

Choose b_j and η_j so that 4.16 holds and

(4.17)
$$\eta_j \geqslant \delta,$$

(4.18)
$$\inf_{\{n\colon |b_j(z_n)| \geqslant \eta_j\}} \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| \geqslant \theta.$$

We have used the fact that $\{z_n\}$ is thin near $\mathfrak{M}(B)$ to get (4.18) and the condition $\sigma(z) \to \infty$ as $z \to \mathfrak{M}(B)$ to get (4.16). We can assume $b_{j-1} \mid b_j$, since we can replace b_j by $b_{j-1}b_j$ if necessary. Letting b_j play the role of f in Lemma 4.8, and using $|b_{j-1}(z)| \geqslant |b_j(z)|$, we see that (4.15) holds. This completes the induction.

Next set

$$\mathcal{R}_j = \{I \in \mathcal{R}_j^0\colon \exists z_n \in T_I, |b_{j+1}(z_n)| \leqslant \eta_{j+1}\}, \quad \mathcal{B}_j = \mathcal{B}_j^0.$$

Since $\mathcal{R}_j \subseteq \mathcal{R}_j^0$, (4.15) implies $\min(D_{\mathcal{R}_j}(L), D_{\mathcal{B}_j}(L)) < \mu_j$ for all arcs L. Let u_j be a function satisfying (4.10)–(4.14) with respect to $\mathcal{R}_j$ and $\mathcal{B}_j$, and set $u = \Sigma u_j$. We claim that u satisfies the requirements of the theorem.

We first show that $u \in \mathrm{VMO}_B$. Fix $j \geqslant 1$. For every arc $I \in \mathcal{R}_j \cup \mathcal{B}_j$ there is $z \in T_I$ with $|b_{j+1}(z)| \leqslant \eta_{j+1}$. By Remark (2) following Theorem 4.7, $u_j \in \mathrm{VMO}_B$. So $u \in \mathrm{VMO}_B$ because Σu_j converges in BMO norm.

To show that $u(z_n) \to \infty$ as $z_n \to \mathfrak{M}(B)$, we assume $|b_j(z_n)| > \eta_j$ and prove $u(z_n) \geqslant j - 1/2^j - 1$. If I is the dyadic arc with $z_n \in T_I$ then $I \in \mathcal{R}_j^0$ and therefore $I \in \mathcal{R}_i$ for some $i \geqslant j$. Hence $u(z_n) \geqslant I(u) - 1 \geqslant I(u_i) - 1 \geqslant i - 1/2^i - 1 \geqslant j - 1/2^j - 1$, where the first inequality follows from $\|u\|_* \leqslant \lambda$ and the third from (4.11).

Since u is clearly positive it remains to prove $u(z) \leqslant \sigma(z)$. Fix $z \in \mathbf{D}$ and let $N \geqslant 0$ be the largest integer for which $|b_N(z)| \geqslant \eta_N$. Let I be the dyadic arc with $z \in T_I$. Then $I \in \mathcal{B}_j$ for $j \geqslant N+2$, hence $I(u_j) \leqslant 1/2^j$ if $j \geqslant N+2$. For $j \leqslant N+1$ we use the trivial estimate $I(u_j) \leqslant \|u_j\|_\infty \leqslant j$. Summing over j, we obtain

$$I(u) = \sum_{j=1}^{\infty} I(u_j) \leqslant \frac{(N+1)(N+2)}{2} + \frac{1}{2^{N+2}} \leqslant N^2 + 3 \leqslant \sigma(z) - 1.$$

Another application of the estimate $\|u\|_* \leqslant \lambda$ gives $u(z) \leqslant \sigma(z)$, completing the proof of the theorem. □

5. Another construction. We complete the proof of the implication from (3) to (1) in Theorem 1 with the following approximate interpolation result.

THEOREM 5.1. *If $\{z_n\}$ is thin near $\mathfrak{M}(B)$ and $\{\lambda_n\}$ is a bounded sequence of complex numbers, then there is $f \in Q_B$ such that $|f(z_n) - \lambda_n| \to 0$ as $z_n \to \mathfrak{M}(B)$. More precisely, we can find f satisfying*:

(a) $\| f \|_\infty \leq M \sup_j |\lambda_j|$,
(b) $|f(z_n) - \lambda_n| < \delta_n \sup_j |\lambda_j|$,
(c) $M_L(f) \leq \Phi(z_L) \sup_j |\lambda_j|$

where $M, \{\delta_n\}$, and Φ depend only on $\{z_n\}$, Φ and $\{\delta_n\}$ are bounded by constants depending only on $\{z_n\}$, and $\delta_n \to 0$ as $z_n \to \mathfrak{M}(B)$, $\Phi(z) \to 0$ as $z \to \mathfrak{M}(B)$.

PROOF. The proof of this theorem will occupy the rest of this section. It will be clear from our construction that the function f will depend linearly on $\{\lambda_n\}$, although we will not need this fact.

DEFINITION. We will say that a sequence of arcs $\{I_j\}$ is thin near $\mathfrak{M}(B)$ if the sequence of points $\{z_{I_j}\}$ is thin near $\mathfrak{M}(B)$.

We construct f as a linear combination of functions a_n which equal 1 on I_{z_n} and vanish off a suitable fixed multiple $\widetilde{q_n I_{z_n}}$. In the first part of this section we choose q_n so that $q_n \to \infty$ as $z_n \to \mathfrak{M}(B)$, $\{\widetilde{q_n I_{z_n}}\}$ is thin near $\mathfrak{M}(B)$, and certain technical conditions are satisfied. The main construction begins with (5.14).

For future reference we note the following facts about the pseudo-hyperbolic metric.

(5.2) If $z \in \mathbf{D}$, z_1 and z_2 are in S_z, and $p_j \geq 1$ with $p_j(1 - |z_j|) \leq 1 - |z|$, $j = 1, 2$, then $|(z_1 - z_2)/(1 - \bar{z}_1 z_2)| \leq 1 - 1/100 p_1 p_2$; this follows easily from

$$1 - \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|^2 = \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \bar{z}_1 z_2|^2} \geq \frac{1 - |z|}{p_1} \frac{1 - |z|}{p_2} \frac{1}{|1 - \bar{z}_1 z_2|^2}.$$

(5.3) Fix $\varepsilon > 0$. If $z_1, z_2 \in \mathbf{D}$ with $|(z_1 - z_2)/(1 - \bar{z}_1 z_2)| < 1 - \varepsilon$, then $|(z - z_2)/(1 - \bar{z}_2 z)| > |(z - z_1)/(1 - \bar{z}_1 z)|^{2/\varepsilon}$ provided $|(z - z_1)/(1 - \bar{z}_1 z)|$ is sufficiently large; one way to see this is to use the conformal invariance of the pseudo-hyperbolic metric to reduce to the case $z = 0$ and $z_1 > 0$, where it follows easily.

LEMMA 5.4. *Suppose $\{z_n\}$ is thin near $\mathfrak{M}(B)$. Then there are numbers $0 \leq \rho_n < 1$, $0 < \gamma_n < 1$, with $\inf_n \gamma_n > 0$, $\rho_n \to 1$ and $\gamma_n \to 1$ as $z_n \to \mathfrak{M}(B)$, such that whenever $\{\zeta_n\} \subseteq \mathbf{D}$ with $\rho(z_n, \zeta_n) < \rho_n$ for all n we will have $\prod_{m \neq n} \rho(\zeta_m, \zeta_n) \geq \gamma_n$ for all n. In particular, $\{\zeta_n\}$ will be thin near $\mathfrak{M}(B)$.*

PROOF. We inductively define a sequence of neighborhoods U_k of $\mathfrak{M}(B)$, such that $U_k \supseteq U_{k+1}$ and $\mathbf{D} \cap (\bigcap_k U_k) = \varnothing$. Let $a > 0$ be such that $\prod_{m \neq n} \rho(z_m, z_n) > e^{-a/2}$ for all n.

Set $U_1 = \mathfrak{M}(H^\infty)$. For $k \geq 2$, if U_{k-1} has been chosen then choose U_k so that the following statement is true. If $\{\zeta_j\}$ satisfies $\rho(z_j, \zeta_j) < 1 - 1/(k+2)$ and if

$$E = \{z_j : z_j \in \mathbf{D} \setminus U_{k-1}\}, \qquad E' = \{\zeta_j : z_j \in \mathbf{D} \setminus U_{k-1}\},$$
$$F = \{z_j : z_j \in U_k\}, \qquad F' = \{\zeta_j : z_j \in U_k\}$$

then, using the notation b_G for the Blaschke product with zeros G,

$$|b_{E'}(z)| > e^{-a/2^k} \quad \text{when } \rho(z, U_k) < 1 - 1/k, \tag{5.5}$$

$$\inf_{\zeta_j \in F'} \prod_{\substack{\zeta_i \in F' \\ i \neq j}} \rho(\zeta_i, \zeta_j) > e^{-a/2^{k+1}}, \tag{5.6}$$

$$|b_{F'}(z)| > e^{-a/2^k} \quad \text{when } \rho(z, \mathbf{D} \backslash U_{k-1}) < 1 - 1/k. \tag{5.7}$$

Justification. Since $b_E \in B^{-1}$ we can choose U_k so that

$$|b_E(z)| > \exp(-a/(2k+4)2^k)$$

when $\rho(z, U_k) < 1 - 1/k$. We do this, and using (5.3) make $\rho(U_k, \mathbf{D} \backslash U_{k-1})$ large enough so that $|b_{E'}(z)| > |b_E(z)|^{2k+4}$ when $\rho(z, U_k) < 1 - 1/k$. This gives (5.5). By (5.3) and Lemma 4.2 of **[12]**, (5.6) and (5.7) will both be satisfied if $\rho(U_k, \mathbf{D} \backslash U_{k-1})$ and $\inf_{z_j \in U_k} \prod_{z_i \in U_k, i \neq j} \rho(z_i, z_j)$ are sufficiently large. Since $\{z_j\}$ is thin near $\mathfrak{M}(B)$ we can make these quantities as large as we like by shrinking U_k further. This completes the induction.

Now define ρ_j and γ_j by $\rho_j = 1 - 1/k$, $\gamma_j = \exp(-3a/2^k)$ for $z_j \in U_k \backslash U_{k+1}$. We show that if $\{\zeta_j\}$ satisfies $\rho(z_j, \zeta_j) \leq \rho_j$, then

$$\prod_{j \neq m} \rho(\zeta_j, \zeta_m) > \gamma_m. \tag{5.8}$$

For each j define $k(j) = k$ if $z_j \in U_k \backslash U_{k+1}$. Fix m and write $\prod_{j \neq m} \rho(\zeta_j, \zeta_m) = RS \prod_{k \geq k(m)+2} T_k$, where R is the product over those j for which $k(j) \leq k(m) - 2$, S is the product over those $j \neq m$ for which $k(m) - 1 \leq k(j) \leq k(m) + 1$, and for $k \geq k(m) + 2$ T_k is the product over those j for which $k(j) = k$. We have $R > \exp(-a/2^{k(m)})$ by (5.5) with $k = k(m)$, $S > \exp(-a/2^{k(m)})$ by (5.6) with $k = k(m) - 1$, and $T_k > \exp(-a/2^k)$ by (5.7) since when $k \geq k(m) + 2$,

$$\rho(\zeta_m, \mathbf{D} \backslash U_{k-1}) \leq \rho(\zeta_m, \mathbf{D} \backslash U_{k(m)+1}) \leq \rho(\zeta_m, z_m) \leq 1 - 1/k(m) < 1 - 1/k.$$

Multiplying the estimates on R, S, and T_k now gives (5.8). This completes the proof of Lemma 5.4. □

For the rest of this section, $\{z_n\}$ is a fixed sequence which is thin near $\mathfrak{M}(B)$. We will denote I_{z_n} by I_n. A consequence of Lemma 5.4 is

COROLLARY 5.9. *There exist $\sigma_n \geq 1$, $\sigma_n \to \infty$ as $z_n \to \mathfrak{M}(B)$, such that $1 \leq t_n \leq \sigma_n$ implies $\{t_n I_n\}$ is thin near $\mathfrak{M}(B)$. In fact, $\prod_{m \neq n} \rho(z_{t_n I_n}, z_{t_m I_m}) \geq \gamma_n$ where γ_n is as in Lemma* 5.4.

PROOF. By shrinking the neighborhoods U_k in the proof of Lemma 5.4 we can assure that $|z_j| \geq \rho_j$. An easy calculation then shows that $\rho(z_n, z_{t_n I_n}) < \rho_n$ if $1 \leq t_n \leq \sigma_n = (1 + \rho_n)/(1 - |z_n| \rho_n)$, hence the corollary follows from the lemma. □

For any such $\{t_n\}$ we will have

$$\sum_{\{j:\, t_j I_j \subseteq L\}} |t_j I_j| \leq A |L| \quad \text{for all arcs } L. \tag{5.10}$$

Here $A = C \max_j \log(1/\gamma_j)$ is independent of the particular choice of $\{t_n\}$. We now choose a specific such sequence $\{q_n\}$, which should satisfy the following conditions.

(5.11) $$1 \leqslant q_n^2 \leqslant \sigma_n \quad \text{and} \quad q_n \to \infty \quad \text{as } z_n \to \mathfrak{M}(B),$$

(5.12) For any n with $\rho_n > \frac{2499}{2500}$ and for any $j \neq n$, the condition
$$q_j I_j \subseteq \widetilde{\widetilde{q_n I_n}} \quad \text{implies } q_j |I_j| \leqslant |I_n| .$$

(5.13) For any arc L there is at most one n such that
$$\rho_n > \frac{2499}{2500}, \quad I_n \subseteq \tilde{\tilde{L}}, \quad \text{and} \quad q_n^2 |I_n| \geqslant |L| .$$

Both (5.12) and (5.13) follow from Lemma 5.4 provided q_n is sufficiently small in comparision to $1/(1-\rho_n)$. In fact, we can take $q_n = \max(1, [2500(1-\rho_n)]^{-1/4})$. We show (5.13) is then satisfied; the proof for (5.12) is similar. Fix L and suppose I_m and I_n both satisfy the conditions of (5.13). By (5.2),

$$\rho(z_m, z_n) \leqslant 1 - \frac{1}{2500 q_m^2 q_n^2} \leqslant \max\left(1 - \frac{1}{2500 q_m^4}, 1 - \frac{1}{2500 q_n^4}\right) = \max(\rho_m, \rho_n).$$

This contradicts Lemma 5.4.

For each $j = 1, 2, \ldots$ define a function a_j by

(5.14) $$a_j(e^{it}) = \begin{cases} 1 & \text{if } e^{it} \in I_j, \\ \left(\log(3q_j)\right)^{-1} \log \dfrac{3q_j \cdot 2\pi |I_j|}{2|t-\theta_j|} & \text{if } e^{it} \in \widetilde{q_j I_j} \setminus I_j, \\ 0 & \text{if } e^{it} \notin \widetilde{q_j I_j} \end{cases}$$

where $e^{i\theta_j}$ is the midpoint of I_j. Then a_j is $(1, 2/\log(3q_j))$ adapted to $q_j I_j$ and $\|a_j\|_* \leqslant 2/\log(3q_j)$.

We will show that an infinite linear combination of the $\{a_j\}$ with bounded coefficients belongs to VMO_B and then use an appropriate choice of coefficients to prove Theorem 5.1.

LEMMA 5.15. *If $\{\beta_j\}$ is a bounded sequence of complex numbers and $f = \sum \beta_j a_j$, then $f \in \mathrm{VMO}_B$. In fact, $M_L(f) \leqslant \Psi(z_L) \sup_j |\beta_j|$, where Ψ depends only on $\{z_n\}$ and $\Psi(z) \to 0$ as $z \to \mathfrak{M}(B)$.*

PROOF. Write $\mathfrak{Q}\mathfrak{I}$ for the set of all arcs $\{q_j I_j\}$ and let $w_n = z_{q_n I_n}$. Given $\varepsilon > 0$, choose a Blaschke product $b \in B^{-1}$ and $\frac{2499}{2500} < \eta < 1$ such that

(5.16) $$|b(w_n)| > \eta \quad \text{implies } 2/\log(3q_n) < \varepsilon.$$

Then, using Lemma 1.1 and Schwarz's Lemma, choose $0 < \omega < 1$ such that

(5.17) $$|b(z_L)| > \omega \quad \text{implies } D_{\{q_j I_j :\, |b(w_j)| \leqslant \eta\}}(\tilde{\tilde{L}}) < \varepsilon,$$

(5.18) $$|b(z_L)| > \omega \quad \text{implies } \sup\left\{\frac{|L|}{|q_j I_j|} : |b(w_j)| \leqslant \eta \text{ and } L \subseteq \widetilde{\widetilde{q_j I_j}}\right\} < \varepsilon.$$

Fix an arc L. We will show that if $|b(z_L)| > \omega$, then $M_L(f) \leqslant d\varepsilon \sup_j |\beta_j d|$ with d independent of $\{\beta_j\}$ and L; this will prove the lemma.

We can assume $\sup_j |\beta_j| = 1$. Split f into four pieces according to the following decomposition of $\mathfrak{Q}\mathfrak{I}$:

$$\mathfrak{Q}\mathfrak{I}_1 = \{q_jI_j \colon |q_jI_j| > |L| \text{ and } |b(w_j)| \leq \eta\},$$
$$\mathfrak{Q}\mathfrak{I}_2 = \{q_jI_j \colon |q_jI_j| \leq |L| \text{ and } |b(w_j)| \leq \eta\},$$
$$\mathfrak{Q}\mathfrak{I}_3 = \{q_jI_j \colon |q_jI_j| > |L| \text{ and } |b(w_j)| > \eta\},$$
$$\mathfrak{Q}\mathfrak{I}_4 = \{q_jI_j \colon |q_jI_j| \leq |L| \text{ and } |b(w_j)| > \eta\}.$$

Set $f_k = \sum_{\{j\colon q_jI_j \in \mathfrak{Q}\mathfrak{I}_k\}} \beta_j a_j$; then $f = f_1 + f_2 + f_3 + f_4$.

Estimation for f_1. Assume $q_jI_j \in \mathfrak{Q}\mathfrak{I}_1$ and a_j does not vanish identically on L. Since $|b(z_L)| > \omega$ and $|b(w_j)| \leq \eta$, (5.18) implies $|L|/|q_jI_j| < \varepsilon$. Now Lemma 4.3 shows that $M_L(f_1) < CA\varepsilon$.

Estimation for f_2. By (5.17), $D_{\mathfrak{Q}\mathfrak{I}_2}(\tilde{\tilde{L}}) < \varepsilon$; Lemma 4.2 then shows that $M_L(f_2) < CA\varepsilon$.

Estimation for f_3. For $q_jI_j \in \mathfrak{Q}\mathfrak{I}_3$ we have by (5.16) that $\beta_j a_j$ is $(1, \varepsilon)$ adapted to q_jI_j. Lemma 4.3 then gives $M_L(f) < CA\varepsilon$.

Estimation for f_4. From (5.13) we have that with at most one exception, the arcs $q_jI_j \in \mathfrak{Q}\mathfrak{I}_4$ such that $\widetilde{q_jI_j} \cap L \neq \varnothing$ must satisfy $|q_j^2I_j| \leq L$, hence $q_j^2I_j \subseteq \tilde{\tilde{L}}$.

If an exceptional arc q_kI_k exists, then $M_L(\beta_k a_k) \leq \|\beta_k a_k\|_* \leq 2/\log(3q_k)$ by (5.16).

For the nonexceptional arcs we use (5.10) with $t_k = q_k^2$. In fact,

$$\sum_{\substack{q_j^2I_j \subseteq \tilde{\tilde{L}} \\ q_jI_j \in \mathfrak{Q}\mathfrak{I}_4}} |q_jI_j| \leq C\varepsilon \sum_{\substack{q_j^2I_j \subseteq \tilde{\tilde{L}} \\ q_jI_j \in \mathfrak{Q}\mathfrak{I}_4}} |q_j^2I_j| \leq CA\varepsilon |L|,$$

where the first inequality follows from (5.16) and the second from (5.10) and the fact that $q_k^2 \leq \sigma_k$. So $M_L(f_4) \leq C\varepsilon A^2 + \varepsilon$, by Lemma 4.2.

Combining the estimates for f_1, f_2, f_3, f_4 proves Lemma 5.15. □

We now prove Theorem 5.1. Suppose $\{\lambda_j\}$ is a bounded sequence. We will find $\{\beta_j\}$ (which will depend linearly on $\{\lambda_j\}$) such that $f = \sum \beta_j a_j$ satisfies (a), (b) and (c) of the statement of the theorem. We can assume $\sup_j |\lambda_j| \leq 1$.

Renumbering, we may assume that $\{q_jI_j\}$ are listed in decreasing order of size. There is a positive sequence $\{\varepsilon_n\}$, bounded and tending to zero as $z_n \to \mathfrak{M}(B)$, such that for any numbers $\xi_1, \ldots, \xi_{n-1}$ we will have

$$V_{\widetilde{q_nI_n}}\left(\sum_{j=1}^{n-1} \xi_j a_j\right) \leq \varepsilon_n \max_{1 \leq j \leq n-1} |\xi_j|. \tag{5.19}$$

To see this, take $\varepsilon_n = CA^{-1}(1 - \gamma_n)$ with γ_n as in Lemma 5.4. If $j < n$ then $\widetilde{q_jI_j} \cap \widetilde{q_nI_n} \neq \varnothing$ implies

$$|q_nI_n|/|q_jI_j| < C\big(1 - |(w_n - w_j)/(1 - \bar{w}_j w_n)|\big) < C(1 - \gamma_n),$$

so (5.19) follows from Lemma 4.3.

We now define $\{\beta_j\}$ inductively as follows. Set $\beta_1 = \lambda_1$. If $n \geq 2$ and β_j has been defined for $1 \leq j < n$, we set $f_{n-1} = \sum_{j=1}^{n-1} \beta_j a_j$ and define $\beta_n = 0$ if $\varepsilon_n > \frac{1}{4}$ and $\beta_n = \lambda_n - I_n(f_{n-1})$ if $\varepsilon_n \leq \frac{1}{4}$.

Claim. $|\beta_n| \leq 4$ and $\|f_n\|_\infty \leq 3$ for all n.

The proof of the claim is by induction on n. The inequalities are obvious if $n = 1$. Let $n > 1$ and assume the inequalities hold for $n - 1$. Then $|\beta_n| \leq |\lambda_n| + \|f_{n-1}\| \leq 1 + 3 = 4$. If $\varepsilon_n > \frac{1}{4}$ the second inequality follows since $f_{n-1} = f_n$. If $\varepsilon_n \leq \frac{1}{4}$ and $t \in \widetilde{q_n I_n}$, (5.19) implies

$$|f_{n-1}(t) - I_n(f_{n-1})| \leq 4\varepsilon_n \leq 1. \tag{5.20}$$

If we write f_n in the form

$$f_n(t) = [f_{n-1}(t) - I_n(f_{n-1})]a_n(t) + f_{n-1}(t)(1 - a_n(t)) + \lambda_n a_n(t)$$

and use (5.20) and the estimates $\|f_{n-1}\|_\infty \leq 3$, $|\lambda_n| \leq 1$, we obtain

$$|f_n(t)| \leq a_n(t) + 3(1 - a_n(t)) + a_n(t) = 3 - a_n(t) \leq 3.$$

This proves the claim.

We now set $f = \sum_{j=1}^\infty \beta_j a_j = \lim_{n\to\infty} f_n$. Clearly f satisfies (a) of Theorem 5.1 with $M = 3$, and (c) follows from Lemma 5.15. Instead of proving (b) directly we show that $|I_n(f) - \lambda_n| < \delta_n'$, with δ_n' independent of $\{\lambda_j\}$ and $\delta_n' \to 0$ as $z_n \to \mathfrak{M}(B)$; this is equivalent by Corollary 2.4.

Suppose n is such that $\varepsilon_n \leq \frac{1}{4}$. Then $I_n(f_n) = \lambda_n$ by construction. We estimate $(1/|I_n|)\int_{I_n} |f - f_n|\, d\theta/2\pi$ under the additional assumption that $\rho_n > \frac{2499}{2500}$. By (5.12), $j > n$ and $\widetilde{q_j I_j} \cap I_n \neq \varnothing$ imply $|q_j I_j| \leq |I_n|$. Applying Lemma 5.4 with $\zeta_j = w_j$ ($j \neq n$) and $\zeta_n = z_n$, we see that $\prod_{j>n} \rho(w_j, z_n) > \gamma_n$. Hence $D_{\{q_j I_j:\, j>n\}}(\tilde{\tilde{I}}_n) < C(1 - \gamma_n)$. Since $f - f_n$ is a sum of functions (4,8) adapted to arcs $q_j I_j$ with $j > n$, Lemma 4.2 implies $(1/|I_n|)\int_{I_n} |f - f_n|\, d\theta/2\pi < CA(1 - \gamma_n) \to 0$ as $z_n \to \mathfrak{M}(B)$. So we can take $\delta_n' = 10$ if $\varepsilon_n > \frac{1}{4}$ or $\rho_n \leq \frac{2499}{2500}$, and $\delta_n' = CA(1 - \gamma_n)$ otherwise. This completes the proof of Theorem 5.1. □

6. Per Beurling functions for QA_B. In this section we prove the last statement of Theorem 1, by a method due to Varapoulos [**23, 24**].

PROPOSITION. *Let $\{z_n\} \subseteq \mathbf{D}$ be thin near $\mathfrak{M}(B)$. Then there exist functions $\phi_n \in QA_B$, $n \geq 1$, such that $\phi_n(z_k) = \delta_{kn}$ and such that $\sum_1^\infty \lambda_n \phi_n \in QA_B$ whenever $\{\lambda_n\}$ is a bounded sequence of complex numbers.*

PROOF. We have shown in §§3–5 that there exist $M < \infty$ and a bounded function Φ on $\mathbf{D}$ satisfying $\Phi > 0$ and $\Phi(z) \to 0$ as $z \to \mathfrak{M}(B)$, such that for any bounded sequence $\{\lambda_k\}$ there is a function $f \in H^\infty$ satisfying $f(z_n) = \lambda_n$ for all n, $\|f\|_\infty \leq M \sup_n |\lambda_n|$, and $(\int |f - f(z)|^2\, dP_z)^{1/2} \leq \Phi(z) \sup_n |\lambda_n|$ for all $z \in \mathbf{D}$.

Fix a positive integer N and let ω be a primitive Nth root of unity. For each $j = 1, \ldots, N$ choose f_j as above with $f_j(z_k) = \omega^{jk}$. For $1 \leq n \leq N$ set $g_n(z) = \frac{1}{N}\sum_{j=1}^N \omega^{-jn} f_j$ and $h_n = g_n^2$. We claim

$$h_n(z_k) = \delta_{kn} \quad \text{for } 1 \leq k, n \leq N, \tag{6.1}$$

$$\sum_{n=1}^N |h_n(z)| \leq M^2 \quad \text{for all } z \in \mathbf{D}, \tag{6.2}$$

$$\sum_{n=1}^N \int |h_n - h_n(z)|^2\, dP_z \leq 4M^2 \Phi(z)^2 \quad \text{for all } z \in \mathbf{D}. \tag{6.3}$$

To prove (6.1), write

$$g_n(z_k) = \frac{1}{N} \sum_{j=1}^{N} \omega^{-jn} f_j(z_k) = \frac{1}{N} \sum_{j=1}^{N} \omega^{(k-n)j} = \delta_{kn}.$$

To prove (6.2), write

$$\sum_{n=1}^{N} |h_n(z)| = \sum_{n=1}^{N} |g_n(z)|^2 = \sum_{n=1}^{N} \frac{1}{N^2} \sum_{j=1}^{N} \omega^{-jn} f_j(z) \omega^{kn} \overline{f_k(z)}$$

$$= \sum_{j,k=1}^{N} \frac{1}{N^2} f_j(z) \overline{f_k(z)} \sum_{n=1}^{N} \omega^{(k-j)n} = \frac{1}{N} \sum_{j=1}^{N} |f_j(z)|^2 \leq M^2.$$

Finally,

$$\sum_{n=1}^{N} \int |h_n - h_n(z)|^2 \, dP_z = \sum_{n=1}^{N} \int |g_n + g_n(z)|^2 |g_n - g_n(z)|^2 \, dP_z$$

$$\leq 4M^2 \sum_{n=1}^{N} \int |g_n - g_n(z)|^2 \, dP_z$$

$$= 4M^2 \frac{1}{N^2} \sum_{n=1}^{N} \sum_{j,k=1}^{N} \omega^{n(j-k)} \int [f_k - f_k(z)] [\overline{f_j - f_j(z)}] \, dP_z$$

$$= 4M^2 \frac{1}{N} \sum_{j=1}^{N} \int |f_j - f_j(z)|^2 \, dP_z \leq 4M^2 \Phi(z)^2,$$

proving (6.3).

Now letting N tend to ∞ and taking weak limits, we obtain functions $\phi_n \in H^\infty$, $n = 1, 2, \ldots$, satisfying $\phi_n(z_k) = \delta_{kn}$, $\sum_{n=1}^{\infty} |\phi_n(z)| \leq M^2$ all $z \in \mathbf{D}$, and $\sum_{n=1}^{\infty} |\phi_n - \phi_n(z)|^2 \, dP_z \leq 4M^2 \Phi(z)^2$. It clearly follows that

$$\left\| \sum_{n=1}^{\infty} \lambda_n \phi_n \right\|_\infty \leq M^2 \sup_k |\lambda_k|$$

and

$$\int \left| \sum_{n=1}^{\infty} \lambda_n \phi_n - \sum_{n=1}^{\infty} \lambda_n \phi_n(z) \right|^2 dP_z \leq 4M^2 \Phi(z)^2 \sup_k |\lambda_k|^2,$$

hence $\sum_{n=1}^{\infty} \lambda_n \phi_n \in QA_B$ for any bounded sequence $\{\lambda_n\}$. □

7. Completion of the proof. We prove (2) implies (3) in Theorem 1. Let $\{z_n\} \subseteq \mathbf{D}$ be a sequence for which (2) holds. Since $\mathrm{VMO}_B \subseteq \mathrm{BMO}$, a theorem of Garnett **[8]** implies that $\{z_n\}$ is an H^∞ interpolating sequence. Hence there is $A < \infty$ such that $\sum_{z_n \in J_I} (1 - |z_n|) < A |I|$ for all arcs I. Denote I_{z_n} by I_n. We first prove

LEMMA 7.1. *Assume condition* (2) *of Theorem* 1 *holds, and let* $N \geq 1$ *and* $\varepsilon > 0$ *be given. Then there is a neighborhood* U *of* $\mathfrak{M}(B)$ *satisfying the following: if* $z_n \in U$, *then* $\sum_{m \neq n, z_m \in S_{NI_n}} (1 - |z_m|) < \varepsilon |I_n|$.

PROOF. Suppose this is false, i.e. suppose we have $N \geq 1$ and $\varepsilon > 0$ such that for all neighborhoods U of $\mathfrak{M}(B)$ there is $z_n \in U$ such that $\sum_{m \neq n, z_m \in S_{NI_n}} (1 - |z_m|) > \varepsilon |I_n|$.

For $E \subseteq \mathbf{N}^+$ we say that E has property (P) if the following holds.

(7.2) For any neighborhood U of $\mathfrak{M}(B)$ there is $n \in E$ such that $z_n \in U$ and

$$\sum_{\substack{m \neq n \\ z_m \in S_{NI_n} \cap U}} (1 - |z_m|) > \tfrac{1}{2}\varepsilon |I_n|.$$

We will now show that $\mathbf{N}^+$ has property (P). Let U be a neighborhood of $\mathfrak{M}(B)$ and let b_{U^c} be the Blaschke product with zeros $\{z_n : z_n \notin U\}$. Since $b_{U^c} \in B^{-1}$ there are neighborhoods of $\mathfrak{M}(B)$ in which $|b_{U^c}|$ stays arbitrarily close to 1. So there is a neighborhood $V \subset U$ such that $z \in V$ implies that $\sum_{z_n \notin U, z_n \in S_{NI_z}} (1 - |z_n|) < \frac{1}{2}\varepsilon |I_z|$. Choose n with $z_n \in V$ and $\sum_{m \neq n, z_m \in S_{NI_n}} (1 - |z_m|) \geq \varepsilon |I_n|$. Then $z_n \in U$ and

$$\sum_{\substack{m \neq n \\ z_m \in S_{NI_n} \cap U}} (1 - |z_m|) = \sum_{\substack{m \neq n \\ z_m \in S_{NI_n}}} (1 - |z_m|) - \sum_{z_m \in S_{NI_n} \setminus U} (1 - |z_m|)$$
$$> \varepsilon |I_n| - \tfrac{1}{2}\varepsilon |I_n| = \tfrac{1}{2}\varepsilon |I_n|.$$

We now recall a result due to K. Hoffman [**12**, Corollary to Theorem 3.2]: Let A be a Blaschke product with zeros $\{\alpha_n\}$ and define $\delta(A) = \inf_n (1 - |\alpha_n|^2) |A'(\alpha_n)|$. Then A has a factorization $A = A_1 A_2$ such that $\delta(A_j) \geq \delta(A)^{1/2}$, $j = 1, 2$. This result, together with the easy observation that if E has (P) and $E = E_1 \cup E_2$ then either E_1 or E_2 has (P), shows that there are sets E with (P) such that $\inf_{n \in E} \prod_{m \neq n, m \in E} \rho(z_m, z_n)$ is arbitrarily close to 1. So there is E with (P) such that for all $n \in E$,

$$\sum_{\substack{m \neq n \\ m \in E \\ z_m \in S_{NI_n}}} (1 - |z_m|) < \tfrac{1}{4}\varepsilon |I_n|. \tag{7.3}$$

By hypothesis there is $f \in \mathrm{VMO}_B$ such that $f(z_n) = 1$ when $n \in E$ and $f(z_n) = 0$ when $n \notin E$. Let $\eta = \varepsilon/(24AN + \varepsilon)$. There is a neighborhood U_0 of $\mathfrak{M}(B)$ such that $z \in U_0$ implies

$$\frac{1}{|I_z|} \int_{I_z} |f - f(z)| \frac{d\theta}{2\pi} < \eta \quad \text{and} \quad \frac{1}{|2NI_z|} \int_{2NI_z} |f - f(z)| \frac{d\theta}{2\pi} < \eta.$$

Let n be as in (7.2) for the neighborhood U_0. By the "first generation" construction [**9**, Chapter 7] we can choose from the set

$$\{z_m : z_m \in U_0 \cap S_{NI_n}, m \notin E\}$$

a subset $\{z_{m_k}\}$ with $\{I_{m_k}\}$ pairwise disjoint and

$$\sum_k (1 - |z_{m_k}|) > \frac{1}{3A} \sum_{\substack{m \notin E \\ z_m \in S_{NI_n} \cap U_0}} (1 - |z_m|) > \frac{\varepsilon}{12A} |I_n|.$$

We have then

$$\eta > \frac{1}{|2NI_n|}\int_{2NI_n} |f-1|\,\frac{d\theta}{2\pi} \geq \frac{1}{|2NI_n|}\sum_k \int_{I_{m_k}} |f-1|\,\frac{d\theta}{2\pi}$$

$$\geq \frac{1}{|2NI_n|}\sum_k \left(|I_{m_k}| - \int_{I_{m_k}} |f|\,\frac{d\theta}{2\pi}\right) \geq \frac{1-\eta}{|2NI_n|}\sum_k |I_{m_k}| > \frac{\varepsilon(1-\eta)}{24AN},$$

contradicting the definition of η. This completes the proof of Lemma 5.1. □

We now continue with the proof of the theorem. Choose a small $\varepsilon > 0$ and a large N. By Lemma 5.1 there is a neighborhood U of $\mathfrak{M}(B)$ such that

$$\sum_{\substack{m\neq n\\ z_m\in S_{NI_n}}} (1-|z_m|^2) < \varepsilon |I_n| \quad \text{if } z_n \in U.$$

Set $\delta = \inf_{m\neq n}\rho(z_m, z_n)$. Then for $z_n \in U$, we have

$$-\log \prod_{m\neq n} \rho(z_m, z_n)^2 = \sum_{m\neq n} -\log \rho(z_m, z_n)^2$$

$$\leq \frac{1}{1-\delta^2}\log\frac{1}{\delta^2}\sum_{m\neq n}\left[1-\rho(z_m, z_n)^2\right]$$

$$= \frac{1}{1-\delta^2}\log\frac{1}{\delta^2}\left[\sum_{\substack{m\neq n\\ z_m\in S_{NI_n}}} + \sum_{z_n\notin S_{NI_n}}\right]\frac{1-|z_n|^2}{|1-\bar z_m z_n|^2}(1-|z_m|^2).$$

The first sum is clearly bounded by $4\pi\varepsilon$. To estimate the second sum, write it as

$$\sum_{k=0}^{\infty}\sum_{z_m\in S_{2^{k+1}NI_n}\setminus S_{2^kNI_n}} \frac{1-|z_n|^2}{|1-\bar z_m z_n|^2}(1-|z_m|^2). \tag{7.4}$$

One easily checks that for $z \notin S_{2^kNI_n}$ we have $(1-|z_n|^2)/|1-\bar z_m z_n|^2 < C/2^kN$. Together with the fact that $\Sigma_{z_m\in S_I}(1-|z_m|) < A|I|$ for all arcs I, this easily implies that (7.4) is bounded by CA/N. Hence $-\log\Pi_{m\neq n}\rho(z_m, z_n)^2$ is bounded by $C(\varepsilon + 1/N)$, which can be made as small as we like by picking ε small enough and N large enough. This shows that $\{z_n\}$ is thin near $\mathfrak{M}(B)$, and completes the proof of Theorem 1. □

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